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In Search of a Theory of Debt Management

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Abstract

A growing literature integrates debt management into models of optimal fiscal policy. One promising theory argues the composition of government debt should be chosen so that fluctuations in its market value offsets changes in expected future deficits. This complete market approach to debt management is valid even when governments only issue non-contingent bonds. Because bond returns are highly correlated it is known this approach implies asset positions which are large multiples of GDP. We show, analytically and numerically, across a wide range of model specifications (habits, productivity shocks, capital accumulation, persistent shocks, etc) that this is only one of the weaknesses of this approach. We find evidence of large fluctuations in positions, enormous changes in portfolios for minor changes in maturities issued and no presumption it is always optimal to issue long term debt and invest in short term assets. We show these extreme, volatile and unstable features are undesirable from a practical perspective for two reasons. Firstly the fragility of the optimal portfolio to small changes in model specification means it is frequently better for fear of model misspecification to follow a balanced budget rather than issue the optimal debt structure. Secondly we show for even miniscule levels of transaction costs governments would prefer a balanced budget rather than the large and volatile positions the complete market approach recommends. We conclude it is difficult to insulate fiscal policy from shocks using the complete markets approach. Due to the yield curve's limited variability maturities are a poor way to substitute for state contingent debt. As a result the recommendations of this approach conflict with a number of features we believe are integral to bond market incompleteness e.g. allowing for transaction costs, liquidity effects, robustness etc. Our belief is that market imperfections need to be explicitly introduced into the model and incorporated into the portfolio problem. Failure to do so means that the complete market approach applied in an incomplete market setting can be seriously misleading.

Keywords: Complete markets, debt management, government debt, maturity structure, yield curve
JEL Classifications: E43, E62

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1 Introduction

Traditionally the practise of debt management has focused on either minimizing the interest cost of borrowing, supporting short term interest rates set by monetary policy makers or assisting capital markets through providing appropriate amounts of risk free assets and liquidity at key maturities (see Missale (1999) for an excellent survey). A more recent literature focuses on the idea that fiscal policy and debt structure should be *jointly* determined. This approach builds from the insight that a key influence on fiscal policy is the government's ability to offset unexpected fluctuations in government expenditure or revenue by managing the size, composition and value of debt.

This fiscal motivation for debt management raises two important research issues. The first is the degree to which bond markets are characterized by incompleteness - that is the extent to which governments are unable to issue state contingent claims. This issue is important in determining how governments should adjust the level of debt in the face of fluctuations. This question is examined in Marcet and Scott (2009) and Scott (2007) who study the behavior of OECD fiscal policy and conclude that governments are constrained by bond market incompleteness. The second research issue concerns what type of debt governments should issue and in what proportion. In a seminal contribution, Angeletos (2002) outlines what we refer to as a complete market approach to debt management. Under the assumption of a Ramsey planner, who seeks to minimize the deadweight loss arising from distortionary taxation, Angeletos shows i) even if a government only issues non-contingent bonds it can still exploit fluctuations in the yield curve and achieve the complete market outcome and ii) the optimal structure for government debt can be solved for by choosing the maturity structure that supports the complete market allocation for fiscal policy. Using this theory of debt management, and in the case of government expenditure shocks only, Angeletos shows that it is optimal for governments to issue long term debt and invest in short term assets.

Many have interpreted this result as saying that the complete market approach to debt management is the correct paradigm to study issues of debt management. The idea that the optimal portfolio involves the government issuing long maturities is also becoming influential in the literature and receives further support in Barro (1999) and (2003) and Nosbusch (2008). The latter also puts the theory to work and shows how cost minimization may be incompatible with optimal fiscal policy. In our view the complete markets approach is a useful place to start research on these issues but it has significant problems.

As has already been documented by Buera and Nicolini (2004) the magnitude of the positions derived from this complete market approach are extremely large multiples of GDP, in many cases the government should hold 5 or 6 times GDP in privately issued short bonds and issue similar amounts of long bonds. No government in the real world conducts debt management this way, not even approximately: governments rarely hold private bonds[1] and the positions held at each maturity are substantially smaller. Logically this gap between the large positions recommended by the models in Angeletos (2002) and Buera and Nicolini (2004) (henceforth ABN) and those observed in practice could be due to a) preferences and technology in reality being different from the relatively simple endowment model of these papers b) market imperfections matter, especially financial market frictions c) governments do not know exactly the value of some parameters in the economy d) governments may pursue suboptimal policies or be subject to other constraints such as time consistency, etc. Angeletos (2002) suggests that a) is a likely candidate for explaining away the gap[2] but suggests that the qualitative implications of complete markets (to issue long term debt and buy short term bonds) are robust to variations in preferences and technology. Our view is different - we believe that the large positions predicted by this method are a result of not explicitly including the reasons for market incompleteness in the government's optimal taxation problem. In other words that b) and c) are more likely explanations for the discrepancy between the predictions of the complete market approach and the data[3]. We believe that a failure to explicitly incorporate these features into the government's debt management problem may lead to misleading recommendations.

In developing this argument we begin by showing that a) is unlikely to be the reason for the discrepancy. For this purpose we first extend the endowment model of ABN (which we summarise in Section 2) to the case of capital accumulation (Section 3). We find that far from reducing the size of the debt positions the introduction of capital accumulation only exacerbates this problem - positions become even larger but also become very volatile.In an effort to reduce the size of these positions we explore other ways of using a) to close the gap between model predictions and data. Specifically in Section 4 we introduce habits, which Wachter (2006) uses to explain volatility in the yield curve. The size of positions remains large, although they are reduced, and again the optimal positions show substantial volatility and frequently reverse sign. The root cause of these problems

[1] Of course, the governments' portfolios during the current crisis are a counterexample, as many governments have lent extensively to banks. However it is doubtful that the motivation behind these loans is fiscal insurance and governments seem committed to unwinding these positions as soon as possible.

[2] *"However, this disturbing result [of debt holdings exploding to plus and minus infinity] is mostly an artefact of an economy without capital"*

[3] We do not explore in this paper the possibility that it is non-Ramsey behaviour that explains the discrepancy. Battaglini and Coate (2008) is a promising direction for such research whereas we wish to focus on the importance of also specifiying market imperfections.

is that the recommended portfolio positions show extreme sensitivity to variations in both the specification of the model and values of the relevant state variables.

Our analysis shows how this extreme sensitivity causes profound problems for the complete market approach. Firstly this sensitivity makes the recommended positions not only very large but also counterfactually volatile. Secondly the sensitvity is so great that the qualitative implications stressed by Barro (1999), (2003), Angeletos (2002) and Nosbusch (2008) are not robust. Small variations even in the choice of maturities available to the governments for a given specification of the economy can easily reverse the issue long-buy short recommendation. Introducing habits and capital accumulation introduces additional state variables that result in the issue long-buy short conclusion being reversed from period to period. Adding productivity to expenditure shocks also can easily produce the implication that governments issue more short term than long term debt. The sensitivity of the complete market approach is what leads to recommendations of large asset positions, substantial volatility anda lack of qualitative robustness regarding the sign of the positions at each maturity.

At the heart of the complete market approach to debt management is a dominant role for fiscal insurance - issuing bonds in a manner that exploits the covariance of bond returns with fluctuations in the net present value of future primary surpluses so as to minimise tax volatility. It is this dominant role for fiscal insurance alone that produces the excess sensitivity of debt positions. However without any explicit consideration of transaction costs or robustness issues such sensitivity is not necessarily a problem. This then raises the issue of how sensitive the optimality of these positions is to small deviations from the complete market approach. This motivates our analysis in Section 5 where we introduce specific reasons why markets may be incomplete and examine whether the large and volatile positions recommended by the complete market approach are costly in this setting. Firstly we consider the case when governments misspecify various features of the economy (the persistence of shocks, the number of shocks, the discount rate, etc). The sensitivity of debt positions to the specification of the model is such that even for small misperceptions, following the complete market approach can lead to significant welfare losses. We find these losses are sufficiently large that the government would frequently prefer to run a balanced budget and so completely forego the advantages of tax smoothing in order to avoid the costs of incorrect debt management. Further so great is the sensitivity that no robust debt management policies emerge - which maturities perform best depends entirely on the misspecification. The importance of this example is that in misspecifying the economy the government is effecitvely in an incomplete market setting - it cannot achieve the complete market outcome. In other words when one is explicit about the reasons for market incompleteness the complete market approach is no longer optimal and fiscal insurance and tax smoothing concerns disappear. This point is reinforced when we introduce another explicit form of market incompleteness - transaction costs. For minimal levels of transaction costs we find that the government would once more prefer to operate a balanced budget rather than the debt management policies recommended by the complete market approach.

We conclude that a theory of optimal debt management needs to supplement the focus of providing insurance against fiscal shocks with an explicit recognition of the capital market imperfections - such as transaction costs, short selling constraints and liquidity effects that generate the bond market incompleteness in the first place. In Faraglia, Marcet and Scott (2009) we show explicitly how to solve for optimal debt management in the face of market incompleteness. However the focus in this present paper is in showing that whilst the complete market approach appears an attractive way of sidestepping problems of market incompleteness explicit consideration of market frictions is needed to reduce the counterfactual sensitivity it implies. Whilst the main message of our paper is to show the problems caused by this sensitivity on a technical note an additional contribution is to show how to solve for the complete markets approach with capital and habits. Extending the model in this way introduces a number of non-trivial technical issues since the level of bonds is now time-varying. We characterize recursively these positions adapting some results in Marcet and Scott (2009). Further, the model solution for the model with habits is non-standard in a way that, to our knowledge, has not been recognized before.

2 Complete Market Approach to Debt Management

This section essentially outlines the model and results of Angeletos (2002) and Buera and Nicolini (2004), henceforth ABN. In later sections we extend and evaluate this model. In other words we consider in this section a complete market approach to debt management. We examine the full commitment model of Lucas and Stokey (1983) augmented to include a productivity shock and calibrated on US data.

2.1 The Economy

The economy produces a single good that cannot be stored. The agent is endowed with one unit of time that it allocates between leisure and labour. Technology for every period t is given by:

$$c_t + g_t \leq \theta_t \left(1 - x_t\right), \tag{1}$$

where x_t, c_t and g_t represent leisure, private consumption and government expenditure respectively and θ_t represents a productivity shock. We shall refer to this version of our model, with some abuse of terminology, as "the endowment economy"[4].

We assume $h_t \equiv (g_t, \theta_t)$ are stochastic and exogenous and represent the only sources of uncertainty in the model. In every period there is a finite number, N, of possible realizations of these shocks $\overline{h}_n \equiv (\overline{g}_n, \overline{\theta}_n)$, $n = 1, ..., N$. As usual, $h^t = (h_0, h_1, ..., h_t)$ represents the history of shocks up to and including period t. Governments and consumers have full information, that is, all variables dated t are restricted to be measurable with respect to h^t. As is standard, we will suppress the dependence of the endogenous variables on h^t whenever there is no confusion.

Preferences are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) + V(x_t)\right], \tag{2}$$

where $0 < \beta < 1$. For simplicity we assume U and V are strictly increasing and strictly concave in their respective arguments. The government has two instruments to finance government expenditure - a flat tax on labour or issue debt/lend to the consumer.

The case of complete markets using Arrow securities requires the government to issue N distinct contingent bonds at time t, each paying one unit of consumption contingent on $h_{t+1} = \overline{h}_n$ for $n = 1, ...N$. The quantity $b_t\left(h^t, \overline{h}_n\right)$ denotes the amount of government bonds issued in period t that pay one unit of consumption in period $t+1$ if $h_{t+1} = \overline{h}_n$ if realization h^t occurred.

The consumer's budget constraint is:

$$\begin{aligned} c_t\left(h^t\right) + & \sum_{n=1}^{N} q_t\left(h^t, \overline{h}_n\right) \; b_t\left(h^t, \overline{h}_n\right) \\ \leq \quad & \left(1 - \tau_t^x\left(h^t\right)\right) w_t\left(h^t\right)\left(1 - x_t\left(h^t\right)\right) + b_{t-1}\left(h^{t-1}, h_t\right), \end{aligned} \tag{3}$$

for all t and h^t, where $q_t\left(h^t, \overline{h}_n\right)$ is the price in terms of consumption of one bond $b_t\left(h^t, \overline{h}_n\right)$, $\tau_t^x\left(h^t\right)$ is the tax on labour and $w_t\left(h^t\right)$ is the wage earned by the consumer.

Finally, the government faces the constraint:

$$g_t\left(h^t\right) + b_{t-1}\left(h^{t-1}, h_t\right) \leq \tau_t^x\left(h^t\right) w_t\left(h^t\right)\left(1 - x_t\left(h^t\right)\right) + \sum_{n=1}^{N} q_t\left(h^t, \overline{h}_n\right) b_t\left(h^t, \overline{h}_n\right). \tag{4}$$

Both the government and the consumer are subject to No-Madoff-game conditions.

Let c denote the sequence of all consumptions $\{c_0, c_1, ...\}$, and similarly for all other variables. A competitive equilibrium is defined as a feasible allocation (c, x, g), a price system (w, q) and a

[4] Strictly speaking this is an endowment economy augmented with work effort, a Robinson Crusoe economy.

government policy (g, τ^x, b) such that, given the price system and government policy, (c, x) solves the firm's and consumer's maximization problem and also satisfies the sequence of government budget constraints (4).

The optimal Ramsey problem chooses policy by selecting the competitive equilibrium that maximizes (2). As shown, for example, in Chari and Kehoe (1999), this is equivalent to maximizing utility subject to (1) and the constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[c_t U_{c,t} - (1 - x_t) V_{x,t}\right] = b_{-1} U_{c,0}, \tag{5}$$

where b_{-1} is the amount of liabilities inherited by the government in period 0, U_c is the marginal utility of consumption and V_x is the marginal utility of leisure.

2.2 The Complete Markets Approach to Debt Management

Under the assumption of complete markets it is always possible to back out the optimal bond holdings. For any given c, x that satisfy (5) define a sequence of random variables z such that

$$z_t(h^{t-1}, h_t) \equiv E\left(\sum_{s=0}^{\infty} \frac{\beta^s}{U_{c,t}} \left[c_{t+s}\ U_{c,t+s} - (1 - x_{t+s})\ V_{x,t+s}\right] \middle| h^{t-1}, h_t \right) \tag{6}$$

It can be shown that all government budget constraints are satisfied with the given sequence c, x if the government issues in period $t-1$ an amount of debt/credit given by

$$b_{t-1}\left(h^{t-1}, \overline{h}_n\right) = z_t\left(h^{t-1}, \overline{h}_n\right) \tag{7}$$

for every n.

Assume by contrast that the government can only issue bonds that yield non-contingent payoffs at different maturities. ABN show how to use $\{z\}$ to derive the optimal structure of government debt in this case. We call this the complete markets approach to debt management even though it is applied to the case of bonds with a non-contingent payoff.

We assume for now that the number of these maturities equals N (that is the number of possible realizations of the shocks) e.g the government completes the markets. Let b_t^j denote the amount of government bonds issued that pay one unit of consumption with certainty in period $t + j$, and let p_t^j denote the market price of this bond in terms of consumption in period t, both p_t^j and b_t^j are a function of h^t. Assume the maturities are consecutive, that is, there is a bond maturing for each

$j = 1, ..., N$. Moreover, assume that in every period the government buys back the entire stock of outstanding debt, so that the budget constraint of the government is

$$g_t + \sum_{j=1}^{N} p_t^{j-1}\ b_{t-1}^j \leq \tau_t^x\ w_t\ (1 - x_t) + \sum_{j=1}^{N} p_t^j\ b_t^j \tag{8}$$

for all t and h^t, and symmetrically for the consumer, where $p_t^0 \equiv 1$. Equilibrium prices satisfy

$$p_t^j = \beta^j \frac{E_t U_{c,t+j}}{U_{c,t}} \tag{9}$$

ABN prove that if bond prices are sufficiently variable, then one can choose each period a portfolio of maturities $(b_t^1, ..., b_t^N)$ such that

$$\sum_{j=1}^{N} p_t^{j-1}\ b_{t-1}^j \left(h^{t-1}\right) = z_t \left(h^t\right) \tag{10}$$

almost surely, for all t. This can be done because even though bonds issued in $t-1$ are not contingent on the realization of h_t, today's value of last period's debt $\sum_{j=0}^{N-1} p_t^j \left(h^t\right)\ b_{t-1}^j \left(h^{t-1}\right)$ is state contingent due to the fact that bond prices vary with the state of nature h_t.

Consider the special case in which productivity is constant $\theta_t = \bar{\theta}$ and government expenditure follows a two step Markov process taking values $\overline{g}_H > \overline{g}_L > 0$ with probabilities of remaining in the same state π_{HH} and π_{LL}. If $b_{-1}^j = 0$ for $j = 1, 2$ then it is well known that variables dated t in the Ramsey allocation depend only on the shock g_t. Therefore in the Ramsey equilibrium, consumption, prices, etc. take two values, one for each realization of the shock. Formally, $z_t \left(h^{t-1}, \overline{g}_i\right) = \overline{z}^i$, $p_t^1 \left(h^{t-1}, \overline{g}_i\right) \equiv \overline{p}^i$ and so on for $i = H, L$ and for all t. Assuming in addition that $g_0 = \overline{g}_H$ it turns out $\overline{z}^H = 0 < \overline{z}^L$. Under these conditions (10) becomes

$$b_{t-1}^1 \left(h^{t-1}\right) + \overline{p}^i\ b_{t-1}^2 \left(h^{t-1}\right) = \overline{z}^i \quad \text{for } i = H, L\ \forall t \tag{11}$$

The necessary and sufficient condition for this problem to have a unique solution is that $\overline{p}^L \neq \overline{p}^H$ such that

$$\begin{pmatrix} b_{t-1}^1(h^{t-1}) \\ b_{t-1}^2(h^{t-1}) \end{pmatrix} = \begin{pmatrix} 1 & \overline{p}^H \\ 1 & \overline{p}^L \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \overline{z}^L \end{pmatrix} = \begin{pmatrix} \frac{\overline{p}^H \overline{z}^L}{\overline{p}^H - \overline{p}^L} \\ \frac{-\overline{z}^L}{\overline{p}^H - \overline{p}^L} \end{pmatrix} \equiv \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \tag{12}$$

for all t. Therefore in this case the amount of debt issued at each maturity is time invariant and assuming standard utility functions, we have $\overline{p}^H < \overline{p}^L$ so that $B^2 > 0$ and $B^1 < 0$. In other words, the optimal debt management policy is for the government to hold short term assets and issue long term liabilities

2.3 Simulations

As stressed by ABN the one-period ahead variability of long rates $\left(\overline{p}^{H}-\overline{p}^{L}\right)$ is not large (both in canonical DSGE models and the real world) so that (12) implies large positions in B^2 are needed to achieve the complete market outcome and a matching but offsetting large position in B^1. To document this problem we calibrate our model to US data and perform simulations. We assume the utility function:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1}+\eta\frac{x_t^{1-\gamma_2}}{1-\gamma_2}$$

and set $\beta = 0.98$, $\gamma_1 = 1$[5] and $\gamma_1 = 2$. We set η such that the government's deficit equals zero in the non stochastic steady state and use the steady state condition to fix the fraction of leisure at 30% of the time endowment. We assume $b_{-1} = 0$. We borrow Chari's et al. (1991) calibration of the government spending and the technological processes, which they choose to match the average share of government spending, the variance and serial correlation of consumption growth in the US. Assuming a two state symmetric Markov process for government expenditure we have $\overline{g}_i = g^*\left(1+\xi_i\right)$, $i = H, L$ and $\xi_H = 0.07 = -\xi_L$, g^* equals to 25% of GDP in the non stochastic steady state and the transition probabilities are $\pi_{HH}^{g} = \pi_{LL}^{g} = 0.95$. For the technological process we assume $\overline{\theta}_i = \exp(\phi_i)$, $i = H, L$ and $\phi_H = 0.04 = -\phi_L$. The transition probabilities of the symmetric Markov process are $\pi_{HH}^{\theta} = \pi_{LL}^{\theta} = 0.91$. In the simulations we show also the case in which the technological process is more persistent than government expenditure ($\pi_{HH}^{\theta} = \pi_{LL}^{\theta} = 0.98$).

To test the sensitivity of our debt management recommendations we consider a range of simulations including both productivity and expenditure shocks, when only productivity or expenditure is the source of uncertainty and also for different degrees of persistence for the shocks. We show results for transition probabilities $\begin{pmatrix}\pi_{HH} & \pi_{HL}\\ \pi_{LH} & \pi_{LL}\end{pmatrix} = \mu\Delta + (1-\mu)\,I$ where Δ are the calibrated probabilities chosen above and $I = \begin{pmatrix}0.5 & 0.5\\ 0.5 & 0.5\end{pmatrix}$. When $\mu = 1$ shocks have the persistence suggested by Chari's et al. (1991) calibration, when $\mu = 0$ is shocks are i.i.d. and for $\mu \in (0,1)$ we have intermediate levels of persistence. Critical to the size of the debt positions the complete market approach recommends is the volatility of the yield curve so as we change the persistence of the shocks we maintain the unconditional variance to the same calibrated level.

Table 1 reports our simulation results[6]. We quote the unconditional average of the ratio of the

[5] Assuming utility to be logarithmic helps simplify our analysis when we allow for capital accumulation. In this case capital is taxed only in periods 0 and 1 and capital taxes are zero thereafter.

[6] Appendix A provides detailed description of the computational methods used to produce the simulations.

value of debt positions with total output (in other words 7.50 means a position of 750% of GDP on average). In the case with either only government expenditure or productivity shocks the economy is characterized by only two states of the world and so the complete market outcome is attained by issuing only two maturities. In the case where we have both productivity and expenditure shocks we have four possible states of the world and so the complete market approach requires issuing four different maturities. We follow Buera and Nicolini (2004) and choose the maturities issued by minimizing the absolute value of the debt positions.

INSERT TABLE 1

Focusing on only one source of uncertainty we find the qualitative recommendations of Angeletos (2002) hold - governments should issue long term debt and invest in short term assets. In the case of persistent government expenditure shocks the optimal positions are large multiples of GDP (the long term debt issued is more than 7 times GDP). The required positions are large because with persistent productivity shocks fluctuations in the fiscal position (z) are large and, as shown in Table 1, fluctuations in the long term interest rate are small. In the case of i.i.d. expenditure shocks or only productivity shocks (whether they are i.i.d. or persistent) the optimal debt positions are much smaller (although still substantially larger than the debt positions we see for OECD economies). It is when we turn to the model that allows for both shocks that we see clearly the problems noted by Buera and Nicolini (2004). Firstly, the required positions are enormous - the government needs to issue debt at each maturity in amounts that vary between 400% and 16000% GDP. Secondly, although the model still recommends issuing long term debt and investing in short term securities the maturity structure is complex and varies dramatically with small changes in maturity. In the case of intermediate persistence in shocks ($\mu = 0.33$) the government should invest in one period bonds, issue 2 year bonds worth 5900% of GDP and invest in three year bonds worth 16000% GDP.

The final rows of Table 1 show simulation results for an economy with both shocks but where we modify the calibrated parameters to allow for a productivity shock that is more persistent than the government expenditure shock. We find two other areas in which the predictions of the complete market approach are volatile and non-robust. The first is we can reverse the recommendation that governments should issue long term debt and invest in short term assets. Changing the persistence of shocks affects the slope of the yield curve and flips around the size of the positions so that now the government should issue short term debt and invest in long term assets. The reason is that whilst interest rates still rise with adverse expenditure shocks the yield curve is now downward sloping, as short rates are more responsive to temporary shocks than long rates in rational expectations models. The second sign of non-robustness occurs when we remove the option that the government can change the maturities it issues. In particular, in the case of $\mu = 0.333$ the maturity structure that minimizes the absolute positions is 1,2,3 and 29 but if we restrict the government to issue maturities at 1,4,13 and 30 (the maturities that minimize the debt positions in the case of persistent shocks, $\mu = 1$) then the matrix of returns becomes singular up to machine precision and the optimal positions tend to plus and minus infinity (numbers for this case, therefore, are not reported in Table 1). Therefore holding fixed the maturity structure small changes in model specification lead to huge changes in positions.

Therefore in the case of an endowment economy calibrated to US data we find that the complete market approach to debt management i) recommends positions that are large multiples of GDP ii) the size of debt positions varies sharply with small changes in maturity and involves simultaneously both issuing and investing in bonds of adjacent maturities iii) is extremely sensitive to small changes in parameter specifications with no presumption that it is always optimal for the government to issue long term debt and invest in short term bonds[7].

3 Introducing Capital Accumulation

The endowment economy is a useful workhorse model but the magnitude and sensitivity of the debt positions we outlined in the previous section could be an artefact of its simplicity. Therefore in this section we use the complete market optimal tax model of Chari et al (1994) to consider Angeletos' (2002) claim that capital mitigates these problems.

3.1 Complete Markets

Assume there are two factors of production: labour $(1-x)$ and capital k, with output produced through a Cobb Douglas function such that the economy's resource constraint is :

$$c_t + g_t + k_t - (1-\delta)k_{t-1} \leq \theta_t \; k_{t-1}^{\alpha}(1-x_t)^{1-\alpha} = \theta_t \; F\left(k_{t-1}, x_t\right) \tag{13}$$

where δ is the depreciation rate. As before, exogenous shocks are $h = (g, \theta)$. The government now has three policy instruments to finance g: taxes on labour τ^x, taxes on capital τ^k and debt.

[7] We have focused purely on the properties of the debt *structure* implied by the complete market approach. However, another source of mismatch with the data comes from the second order properties of deficits and debt. As shown in Marcet and Scott (2009) the complete market Ramsey outcome implies debt should show i) *less* persistence than other variables and ii) a negative co-movement with deficits. These findings are replicated in our simulations here. However in practice debt shows *greater* persistence than other variables and a positive co-movement of deficit and debt.

For this problem to be of interest it is standard to restrict capital taxes in two ways. First we need to bound the initial period capital tax to prevent the planner from achieving the first best through a capital levy. We therefore add the constraint $\tau_0^k \leq \overline{\tau}^k$ for a fixed constant $\overline{\tau}^k$. We also need to assume that capital taxes are decided one period in advance (see also Farhi (2005)) otherwise debt and taxes in equilibrium would be underdetermined and the role of debt management could be supplanted by state contingent capital taxation (see Chari and Kehoe (1999)). Note that as a result of this assumption we denote by τ_t^k the tax that is applied to capital income in period t even though this tax is set with information on h^{t-1}.

As before, we start with the case of complete markets where the government has full access to a complete set of contingent Arrow-Debreu securities. The consumer's budget constraint is:

$$\begin{aligned} c_t\left(h^t\right)+k_t\left(h^t\right)+\sum_{n=1}^{N} q_t\left(h^t,\overline{h}_n\right)\ b_t\left(h^t,\overline{h}_n\right) \leq & \left[\left(1-\tau_t^k\left(h^{t-1}\right)\right) r_t\left(h^t\right)+1-\delta\right] k_{t-1}\left(h^{t-1}\right) \\ & +\left(1-\tau_t^x\left(h^t\right)\right) w_t\left(h^t\right)\left(1-x_t\left(h^t\right)\right)+b_{t-1}\left(h^{t-1},h_t\right) \end{aligned}$$

and the government's:

$$\begin{aligned} g_t\left(h^t\right)+b_{t-1}\left(h^{t-1},h_t\right) \leq & \ \tau_t^k\left(h^{t-1}\right)\ r_t\left(h^t\right)\ k_{t-1}\left(h^{t-1}\right) \\ & +\tau_t^x\left(h^t\right)\ w_t\left(h^t\right)\ \left(1-x_t\left(h^t\right)\right)+\sum_{n=1}^{N} q_t\left(h^t,\overline{h}_n\right)\ b_t\left(h^t,\overline{h}_n\right) \end{aligned}$$

where r_t denotes the rental price of capital.

The set of constraints in the Ramsey problem is now augmented with the consumer's Euler equation with respect to capital, viz.,

$$U_{c,t}=\beta E_t\left\{U_{c,t+1}\left[\left(1-\tau_{t+1}^k\right) r_{t+1}+1-\delta\right]\right\}. \tag{14}$$

Firms' maximization implies $r_t = F_{k,t}$, $w_t = F_{l,t}$.

The results of Chari and Kehoe (1999) guarantee that the implementability constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t\left[c_t U_{c,t}-(1-x_t) V_{x,t}\right]=U_{c,0}\left[(\left(1-\tau_0^k\right) F_{k,0}+1-\delta) k_{-1}+b_{-1}\right]$$

plus the feasibility constraint (13) are necessary and sufficient conditions for a competitive equilibrium.

For given sequences c, k, x that satisfy these conditions we can build the expected discounted sum of future surpluses of the government in each period z:

$$z_t^k\left(h^{t-1}, h_t\right) \equiv E\left(\sum_{s=0}^{\infty}\beta^s\left[c_{t+s}\frac{U_{c,t+s}}{U_{c,t}} - (1-x_{t+s})\frac{V_{x,t+s}}{U_{c,t}}\right]\Bigg| h^t\right) - \left[\left(1-\tau_t^k\left(h^{t-1}\right)\right)F_{k,t}\left(h^t\right)+1-\delta\right]\ k_{t-1}\left(h^{t-1}\right). \quad (15)$$

Once more we can solve for the optimal portfolio by using (7) for each feasible c, k, x. However, unlike the case for the endowment economy the optimal bond positions are no longer constant. Chari et al. (1994) show that the Ramsey solution to this problem satisfies the recursive structure:

$$\left[k_t, c_t, x_t, \tau_t^x, \tau_t^k\right]' = G(h_t, k_{t-1})$$

for all $t \geq 1$ for some time-invariant function G. Using Proposition 1A in Marcet and Scott (2009) this implies the existence of a time-invariant function D such that

$$D(k_{t-1}, \overline{h}_n) = z_t^k(h^{t-1}, \overline{h}_n)$$

for all $t \geq 1$, all h^t all n and the z_t^k obtained when plugging the optimal solution in (15). In other words, even though $z_t^k(h^{t-1}, \overline{h}_n)$ potentially depends on all past shocks these are effectively summarized by the previous period capital stock in the optimal solution. Therefore using (7) we have that the Ramsey optimum for debt under complete markets is $b_{t-1}(h^{t-1}, \overline{h}_n) = D(k_{t-1}, \overline{h}_n)$. The result of adding capital is that the contingent bond positions that complete the market are no longer constant but are a function of the capital stock.

3.2 The Complete Market Approach to Debt Management

We now turn to the standard debt management case where the government issues debt that pays a fixed amount at the time of maturity. We assume the government issues N consecutive maturities. The government can effectively complete the markets if it can find bond holdings b_{t-1}^j for each maturity such that

$$\sum_{j=1}^{N} p_t^{j-1}(h^{t-1}, \overline{h}_n)\ b_{t-1}^j\left(h^{t-1}\right) = D(k_{t-1}(h^{t-1}), \overline{h}_n) \quad (16)$$

for all t, all h^t and all n. This gives N equations to solve for the unknowns $(b_{t-1}^1\left(h^{t-1}\right), ..., b_{t-1}^N\left(h^{t-1}\right))$ in each period. Furthermore, since the recursive structure of the Ramsey solution implies $P^n\left(k_{t-1}(h^{t-1}), \overline{h}_j\right) \equiv p_t^n(h^{t-1}, \overline{h}_j)$ for N time-invariant functions P^n, for all $t \geq 1$, all h^{t-1} and all $n, j = 1, ..., N$, this gives a recursive solution for the optimal bond portfolio. More precisely, letting $\Pi : R_+ \to R^{N\times N}$ be defined as

$$\Pi(k) \equiv \begin{bmatrix} 1 & P^1\left(k, \overline{h}_1\right) & ... & P^{N-1}\left(k, \overline{h}_1\right) \\ \vdots & & & \vdots \\ 1 & P^1\left(k, , \overline{h}_N\right) & ... & P^{N-1}\left(k, , \overline{h}_N\right) \end{bmatrix} \tag{17}$$

and assuming $\Pi(k_t)$ is an invertible matrix with probability one,[8] then the time-invariant function $B : R_+ \to R^N$ defined by

$$\begin{bmatrix} b^1_{t-1} \\ \vdots \\ b^N_{t-1} \end{bmatrix} = \left[\Pi(k_{t-1})\right]^{-1} \begin{bmatrix} D(k_{t-1}, \overline{h}^1) \\ \vdots \\ D(k_{t-1}, \overline{h}_N) \end{bmatrix} \equiv B(k_{t-1}) \tag{18}$$

gives the portfolio that effectively completes the markets for all $t \geq 1$, all h^t.[9] Therefore, with capital accumulation the amount issued of maturity j at time t is no longer constant but is now a time-invariant function of current capital.

The formula (18) already gives a strong hint that the resulting bond positions are likely to be very volatile positions. This is because, as can be seen from the definition of $\Pi(k)$ in (17), each row of $\Pi(k)$ contains the yield curve conditional on each realization of the shock. Roughly speaking, the yield curve between period t and period $t + 1$ jumps from one row to another row in $\Pi(k)$. As is well known, both in the model and in the real world the yield curve does not change much from one period to the next, therefore for any realistic calibration the rows of $\Pi(k)$ are quite similar and, as a consequence, the matrix $\Pi(k)$ is likely to be nearly singular. Near singularity means that small changes in k are likely to bring about large changes in $[\Pi(k)]^{-1}$, since this inverse is nearly ill-defined. When capital moves through time $\Pi(k)$ moves around this singularity, and the bond positions which depend on $[\Pi(k_{t-1})]^{-1}$ are likely to have very large movements through time.

3.3 Simulations

Table 2 summarizes the results for simulations of the model with capital accumulation. We set $\alpha = 0.4$, the depreciation rate $\delta = 0.05$, assume that the initial value of government debt is always zero and set the initial capital stock equal to its deterministic steady state value in the Ramsey

[8] The "probability" statement is with respect to the distribution on k_t induced by the Ramsey solution. If $\Pi(k_t)$ is singular with positive probability then, quite simply, the complete markets approach can not be implemented with N maturities.

[9] Notice that the Ramsey solution is only fully recursive for $t \geq 1$, because variables such as consumption or capital are only time-invariant functions after period 1, but the portfolio that completes the markets turns out to be time-invariant for $t \geq 0$.

allocation[10]. As the bond holdings issued in each period are no longer constant we report both the average structure of the value of debt and also the average of the 5% lowest and 5% highest positions for each maturity, so as to indicate the volatility of the positions. The details of how we obtain the simulations for this and all the models solved in this paper are given in appendix A. This appendix gives a step-by-step account of how each policy function is computed and how the discounted budget constraints are insured. In summary, we find an optimal policy G and the expected discounted sum D such the FOC optimality conditions are satisfied. Functions G and D are found by standard Parameterrized Expectation Algorithm (PEA). Given G we then find P by approximationg the expectations of future marginal utilities in one step and construct Π.

INSERT TABLE 2

The results show that adding capital accumulation only exacerbates the magnitude of the positions. When we allow for capital accumulation we allow another margin through which agents can smooth consumption and so interest rates and bond prices are less volatile requiring larger positions to achieve the complete market outcome. As we explained above capital accumulation also makes the optimal debt positions time varying and Table 2 shows that the required variation is substantial. For instance, in the case of persistent productivity and expenditure shocks although on average the government issues long term debt worth 3344% of GDP in 5% of the periods it issues long term debt worth on average around 1594% GDP and at the other extreme in 5% of periods issuance averages around 6629%.

We also find another dimension in which the qualitative recommendation of the complete markets approach to debt management is undermined. Let us go back to the case where the economy is perturbed only by a persistent productivity shock and then use the complete market approach to solve for the optimal debt positions when the government issues a one period bond and a j-period bond, $j = 2, ..., 30$. Any one of these j's is sufficient to complete the market given in this case there are only two shocks. The optimal bond positions are shown in Figure 1 as a function of each possible j. For $j < 18$ the government should issue short term debt and invest in long term bonds. However when the government is constrained to issue long term bonds of maturity 18 or greater than the result flips around and now issuing long term debt and investing short term becomes optimal. The notion that optimal portfolio structure can change so dramatically depending on whether

[10] More precisely, we consider the deterministic steady state when g_t, θ_t are equal to the constants g^*,θ^*, there are no capital taxes and labour taxes are constant.

the government issues a 17 or 18 year bond seems both an undesirable property and worrying from a policy perspective[11].

Further evidence of the sensitivity of the model to small changes in specification is shown in Figure 2 which plots the policy function for the case where the government has access to markets for 1-period and 16-period bond. Figure 2 shows that there is a value of capital k^* close to 1291 such that, if $k_t < k^*$ the government should issue short term debt and invest in long term assets, and these signs are reversed for $k_t > k^*$. Furthermore, long bonds converge to plus (minus) infinity if $k_t \nearrow k^*$ ($k_t \searrow k^*$). The reason for this policy function is that the matrix of bond returns $\Pi(k^*)$ is non-invertible. More precisely, we see from (18) that if the shock takes only two values

$$b_{t-1}^{16} \equiv B^N(k_{t-1}) = \frac{D(k_{t-1},\theta^L) - D(k_{t-1},\theta^H)}{P^{15}(k_{t-1},\overline{\theta}^L) - P^{15}(k_{t-1},\overline{\theta}^H)}$$

It turns out that the denominator in the above expression is zero at $k_{t-1} = k^*$, negative (positive) for lower (higher) k_{t-1}. The numerator, however, is never close to zero. This singularity is the reason for the change in sign and the asymptotes in Figure 2. In our simulations the probability of capital being less than k^* in the steady state distribution is 49.8%, hence the singularity occurs at a level of capital close to the median. Therefore, the probability of seeing an asset position that changes dramatically from one period to the next in a given realization is very high and the time series volatility of bond positions is very large. Note that a country that persisted every period in issuing long term debt and investing in short term bonds would actually lead to excess volatility in taxes. In other words, far from confirming the qualitative insights of Angeletos (2002) the addition of capital accumulation significantly undermines the recommendation to always issue long and buy short[12].

INSERT FIGURE 1 AND FIGURE 2

4 Habits and Term Structure Volatility

In this section we introduce habits into the utility function of the consumer. We have two purposes in mind. First we want to study the robustness of the complete markets approach to this commonly

[11] Given that emerging markets often only have access to bonds of less than 10 year maturity Figure 1 suggests that the complete market recommendation for emerging markets is the reverse of that to OECD economies. Emerging markets should issue short and buy long.

[12] If we pursue the theme of emerging markets the implication of Figure 2 is that countries should pursue the opposite of Angeletos recommendation e.g they should issue short and invest long, along their development path but as they approach their steady state debt management will show dramatic reversals from period to period.

used utility function. Habits have been widely used as a means of matching asset market puzzles in the literature e.g. Constantinides (1990), Campbell and Cochrane (1999), Wachter (2006). In essence it makes interest rates a function of consumption growth and so the slope of the yield curve depends on the rate of change of consumption growth and so raises the volatility of both. Second and, perhaps more importantly, is to use the model with habits to match relevant aspects of the volatility of the yield curve.

Relating the model to the volatility of the yield curve is important because one potential criticism of our findings of extreme, volatile and unstable positions in Sections 2 and 3 is that they are based around models which produce counterfactually low volatility in the slope of the yield curve. To understand the importance of the volatility of the yield curve to the size of the positions recommended by the complete market approach consider again the simple model of Section 2 when g can take two possible values $\overline{g}_H, \overline{g}_L$ and assume the government issues a short bond that matures in S periods ($S < M$). In this case markets would be effectively completed by a portfolio b_t^S, b_t^M satisfying

$$\overline{p}_t^{S-1,i} b_{t-1}^S + \overline{p}_t^{M-1,i} \, b_{t-1}^M = \overline{z}^i \quad \text{for } i = H, L \quad \text{for all } h^{t-1}$$

where $\overline{p}_t^{S-1,H}$ is shorthand for $\overline{p}_t^{S-1}(h^{t-1}, \overline{g}_H)$ and so on. Using the complete market methodology gives the following optimal long position

$$b_{t-1}^M = \frac{\overline{p}_t^{S-1,H} \; \overline{z}^L - \overline{p}_t^{S-1,L} \; \overline{z}^H}{\overline{p}_t^{S-1,H} \; \overline{p}_t^{M-1,L} - \overline{p}_t^{S-1,L} \; \overline{p}_t^{M-1,H}}$$

The closer to zero is the denominator $[\overline{p}_t^{S-1,H} \; \overline{p}_t^{M-1,L} - \overline{p}_t^{S-1,L} \; \overline{p}_t^{M-1,H}]$ then *ceteris paribus* the larger is the absolute value of b_{t-1}^M. Log-linearizing this denominator around 1 and rearranging gives:

$$(S-1)(spr_t^H - spr_t^L) + (M-S)(r_t^{M-1,H} - r_t^{M-1,L})$$

where $spr_t^i \equiv r_t^{M-1,i} - r_t^{S-1,i}$ is the interest rate spread between long and short bonds for realizations $i = H, L$, and r is the annualized net interest rate at each maturity for each given realization of the shock. The terms $\left(spr_t^H - spr_t^L\right)$ and $(r_t^{M-1,H} - r_t^{M-1,L})$ are closely related to the variability of the spread and the return on an M period bond conditional on information up to $t-1$[13]. Therefore the greater the volatility of the spread conditional on past information, and the larger the one period ahead volatility of the return on the M period bond, the smaller the required optimal position

[13] More precisely, in the case of a symmetric distribution where $\Pr ob_{t-1}(g_t = g^i) = .5$ it is easy to check that $E_{t-1}|spr_t| = spr_t^H - spr_t^L$.

to complete the market. Introducing habits is therefore an important way of changing the model specification to reduce the size of positions recommended by the complete market approach.

With habits in consumption the utility function of the consumer becomes:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t, c_{t-1}\right) + V\left(x_t\right) \right], \tag{19}$$

The resource and budget constraints are the same as in the endowment model developed in Section 2. Equilibrium prices are given as before by (9) where marginal utility of consumption is now given by:

$$U_{c,t} \equiv \frac{\partial\, U\left(c_t, c_{t-1}\right)}{\partial c_t} + \beta E_t \left[\frac{\partial\, U\left(c_{t+1}, c_t\right)}{\partial c_t} \right] \tag{20}$$

The marginal utility now depends on past, current and expected future levels of consumption because of the presence of internal habits. The implementability condition is (5) as in section 2 but with $U_{c,t}$ given by the above formula.

The presence of future variables in $U_{c,t}$ introduces some technical difficulties and non-standard aspects in the optimal policy. Unlike the case of section 2 the first order conditions of the Ramsey policy include some intertemporal terms since c_t now appears in $U_{c,t}, U_{c,t-1}$ and $U_{c,t+1}$. Further in order to write recursively the problem we need to operate on the implementability constraint until we can express the Lagrangian as a recursive sum from period 1 onwards. It turns out that the policy function is time-invariant $c_t = G(h_t, c_{t-1})$ for $t \geq 1$, but it is a different function in period zero.[14] This different policy in period zero is usually avoided in endowment models by assuming zero initial debt, our point is that even with zero initial debt the policy function in $t = 0$ is different. To our knowledge these issues had not been dealt with in the literature, they are all carefully addressed in the appendix.

In the appendix we also discuss how to compute the policy function G (obviously different from the function in the previous section) and how to separately compute the solution for period 0. Using a similar argument as in the last section, we conclude

$$\left[b_t^1, ..., b_t^N\right]' = B(c_t) \tag{21}$$

for some time-invariant function B and all h^t for all $t \geq 1$. Thus, the level and composition of debt that effectively complete the markets now varies with current consumption. The formula for B is

[14] Some of these technical difficulties would be avoided by assuming external habits. There is a literature that studies how optimal fiscal policy can be used to treat the externality that arises from the spillovers of others' consumption caused by the external habits. See, for example, Ljungqvist and Uhlig (2000), Alonso-Carrera, Caballé and Raurich (2004) and references therein. We chose internal habits to avoid dealing with issues of externalities.

obtained, analogous to the model for capital, by constructing a matrix $\Pi(c_t)$ with the yield curve for each realization in each row and inverting this matrix for each c_t.

In our simulations we assume the functional form

$$U\left(c_t, c_{t-1}\right) + V\left(x_t\right) \equiv \frac{\left(c_t - \chi c_{t-1}\right)^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2},$$

The additional parameter introduced is χ and we can choose this so that our model matches key features of the data. Unfortunately, to our knowledge, there does not exist a full general equilibrium model with habits which adequately captures the stochastic features of the yield curve at all maturities. Motivated by the analysis above we therefore focus instead on matching the volatility of the spread and the M period return only. To do so we estimate the one step ahead forecast error variance of $x_t = spr_t + 9r_t^{10}$ (where the spread is that between the ten and one year US bond) over the period 1949 to 2004. Applying standard model selection criteria on a VAR of lags and a set of related macroeconomic variables (e.g. GDP growth, interest rates, primary deficit, inflation) yields an equation of the form

$$x_t = \alpha_1 + \alpha_2 x_{t-1} + \alpha_3 \pi_t + +\alpha_4 \pi_{t-1} + \varepsilon_t$$

The variance of ε is our measure of $var_{t-1}\left(x_t\right)$ which leads to an estimate of $\frac{\sqrt{E(var_{t-1}(\varepsilon_t))}}{Er_t}$ equal to 3.38. Comparing this to our model simulations of the previous sections confirms how poorly they perform in terms of producing volatility in the yield curve. For instance, in the model without capital the conditional volatility of the spread between one and 10 period bonds, is equal to only 0.052 for the model with government spending shocks, 0.317 for the model with technology shocks and only 1.172 even if we allow for both shocks. As we show below, in the case of habits, no capital and productivity shocks alone we are able to match exactly the volatility of x_t although for the case of expenditure shocks only or both expenditure and productivity shocks we once again fail to match fully the volatility. For these cases we therefore calibrate our model differently and focus on matching just the volatility of the spread between the ten year and one year rate. A similar model selection procedure as that described above gives a forecasting equation of the form :

$$spr_t = \alpha_1 + \alpha_2 spr_{t-1} + \alpha_3 \frac{def_{t-2}}{gdp_{t-2}} + \alpha_4 r_{t-2} + u_t$$

where $\frac{def_t}{gdp_t}$ is the primary deficit/GDP ratio and r_t is the one year real interest rate. The variance of u is our measure of $var_{t-1}\left(spr_t\right)$ which leads to an estimate of $\frac{\sqrt{E(var_{t-1}(u_t))}}{Er_t}$ equal to 0.341

(comparing this with our model simulations gives 0.03, 0.044 and 0.163 for the case of no habits and just g shocks, productivity shocks and both expenditure and productivity shocks).

INSERT TABLE 3

Table 3 summarizes the results of our simulations using the calibrated values of χ and also comparing with the case $\chi = 0$. In order to maximize the volatility of interest rates we only show results for persistent shocks. Because the introduction of habits raises the volatility of the interest rate spread it does lower the magnitude of the positions. However, although the magnitude of positions is reduced they remain large (for instance in the case with both shocks the government has to issue 22 year bonds to the value of 11.48 times GDP and invest in 10 year bonds worth 18.23 times GDP). Whilst allowing for habits attenuates the size of the required positions it also creates a substantial additional problem. Increasing the volatility of interest rates and the term spread reduces the average size of the positions but at the expense of substantially increasing their volatility. For instance, if we focus on the higher 5% realizations of the long bond issuance they are on average 99.10 times GDP, while if we focus on the lowest 5% realizations we find that the government invests in 22 year bonds to the value of 62.69 GDP on average in this interval. Therefore the complete market approach recommends hugely volatile positions and, once again, the simple qualitative recommendation of issuing long term bonds and investing in short term assets is easily overturned since the government invests heavily in long maturities in many periods.

The reason behind these results is shown in Figure 3 which reports the policy functions for the value of the bond positions as a function of consumption.[15] The policy functions for bonds of 10, 15 and 22 period maturity show a spike at the same level of consumption. At this level of consumption the matrix of returns is non-invertible and at this point the sign of the bond holding switches. The reason for this behavior is analogous to the one described for the model with capital, namely, the singularity of the matrix $\Pi(c_t)$ for some value of consumption. Therefore for only small changes in consumption we see an enormous shift in debt positions with long term debt going from large negative values to large positive ones. This reversal of optimal debt management occurs despite the fact that interest rates do rise in response to adverse expenditure shocks - a combination that Angeletos (2002) and others stress as important for making it optimal for governments to issue long term debt.

[15] Four lines appear in each graph of Figure 3, each line for each current realization of the shocks. This may seem odd given our previous observation that the position of the bonds is determined by current consumption. But what is reported in the Figure is the *value* of the bond, which is multiplied by the price and, therefore, contingent on today's realization of the shocks.

As stated earlier our model does not fully capture all aspects of the yield curve. Therefore it could be argued that when a proper general equilibrium model of the term structure is developed the complete market approach would predict positions that better match with the observed practice of debt management. However this section suggests that even if a more volatile term structure would help produce more modest positions it would only worsen the fit in another dimension. Greater term structure volatility reduces the size of the positions but substantially increases their volatility.

INSERT FIGURE 3

5 Robustness

Previous sections have shown how the positions the complete market approach advocates are extreme and "unreasonable" when compared with actual debt management practice. For instance, we have shown how the complete market approach leads to positions which are large multiples of GDP and which in the presence of habits or capital accumulation show high volatility. We have also shown how variations in the parameterisations of the economy lead to substantial changes in the optimal portfolio structure such that there is no presumption that it is optimal to issue long term debt and invest in short term assets. However within the context of the framework we have been using the size, sensitivity and volatility of the bond positions cannot be used as a criticism of the complete market approach - given the planner's knowledge of the environment and the absence of transaction costs these are the optimal positions. Pointing to extreme magnitudes or volatility cannot be a justified criticism unless these positions come with some cost. It is to this topic we now turn.

In this section we extend our analysis to consider how robust these portfolios are to alternative specifications. Firstly we focus on the case where the government incorrectly specifies the nature of the economy (variously the persistence of shocks, the number of states of the world and the discount rate). We show how relatively small misspecifications can lead to large welfare costs in pursuing the complete market recommendations such that governments are often better pursuing a balanced budget outcome. Secondly we consider the size of transaction costs necessary to offset the insurance benefits of the complete market approach to debt management. We show how given the size of the positions the complete market approach advocates even *de minimus* transaction costs make these inferior to a balanced budget approach. Introducing model misspecification and transaction costs means we are in a world of incomplete markets. It is hardly surprising therefore that welfare falls relative to the first best complete market outcome. However the point of this section is not to note that welfare falls but that the potential costs are so large that governments would frequently prefer to run a balanced budget and forego any form of debt management. In other words once market incompleteness is introduced explicitly into the model the complete market approach is far from optimal and not even an approximate guide for policy.

5.1 Government Misperceptions

Key to the size of the positions that the complete market approach recommends is the persistence of the shocks. Errors in perceiving persistence of shocks will therefore translate into sub-optimal portfolio positions and given the sensitivity of positions there is potential for these errors to be large. To evaluate the welfare costs of these errors we use :

$$R(\rho, \overline{\rho}) = \frac{W_X - W_{BB}}{W_{CM} - W_{BB}}$$

where W_i denotes the welfare level obtained for an economy where $i = CM, BB$ and X and CM denotes the complete market approach when the government correctly specifies the economy, BB is when the government runs a balanced budget every period and X is an economy in which the government believes that the primitives of the economy are given by vector $\overline{\rho}$ whereas in practice they are given by $\boldsymbol{\rho}$. The ratio $R(.,.)$ captures the proportion of the gains of optimal debt issuance that are preserved when the government misperceives the economy. The denominator measures the maximal welfare gains that come from issuing debt and so $R(.,.)$ is bounded from above by 1. For values of $R(.,.)$ between 0 and 1 the misspecification of the primitives reduces the welfare gains from debt management but still leads to an improvement over the balanced budget case. In the case when $R(.,.) < 0$ then attempts at optimal debt management are actually producing worse outcomes than a balanced budget and no debt issuance.

In constructing this measure we are effectively using the balanced budget case as a benchmark.Another possible benchmark would be incomplete markets when the government can only issue one period bonds e.g Marcet and Scott (2009). In this case the question of how much debt to issue and at what maturities become one and the same thing. However we prefer to use the balanced budget outcome as a benchmark. Given that it must be the case that $W_{IM,1}$ (the welfare reached under incomplete markets with just one period debt) must exceed W_{BB} then negative values of $R(.,.)$ are less likely using balanced budgets as a benchmark. In other words we are giving complete markets a better chance. Using W_{BB} rather than $W_{IM,1}$ also makes the numerator and denominator larger and reduces the sensitvity of $R(.,.)$ and also avoids possible approximation errors given we cannot solve exaclty for the incomplete market case.

5.1.1 Misperceiving Persistence

Consider the earlier model of Section 2.3 of an economy without capital accumulation and subject only to government expenditure shocks. Government expenditure can take only two values, high and low, and there exists a constant Markov transition matrix $\begin{pmatrix} \pi_{HH} & \pi_{HL} \\ \pi_{LH} & \pi_{LL} \end{pmatrix}$. Assume our earlier calibration that $\pi^g_{HH} = \pi^g_{LL} = 0.95$ but that the government has beliefs $\widetilde{\pi}^g_{HH}, \widetilde{\pi}^g_{LL}$ where $\widetilde{\pi}^g_{HH} \neq \pi^g_{HH}$ and $\widetilde{\pi}^g_{LL} \neq \pi^g_{LL}$. . Assuming the government can issue only one year and thirty year bonds we show in Figure 4a how $R(\pi, \widetilde{\pi})$ varies as beliefs alter with respect to reality. So long as governments underestimate the persistence of the process $R(.,.)$ is always postive even if less than 1. The worse the underestimate of persistence the more that $R(.,.)$ tends to the balanced budget benchmark. However in the case that the government overestimates persitence (and so takes larger positions) then $R(.,.)$ drops rapidly and quickly turns negative and takes substantial values. Governments are better following a balanced budget than operating complete market policies when they overestimate the persistence of government expenditure shocks. Figure 4b shows the same experiment but around the actual probabilities $\pi^g_{HH} = \pi^g_{LL} = 0.5$. The same pattern emerges - underestimating persistence leads to welfare declines relative to the true complete market outcome but the costs increase only slowly with the level of underestimation. However once again the welfare costs increase dramatically in the case of overpersistence and lead to even greater losses than Figure 4a.

INSERT FIGURE 4A AND 4B

As the precise positions recommended by the complete market approach are very sensitive to the choice of maturities the government issues so too will be the welfare losses. Table 4 investigates this by calculating $R(.,.)$ across all combinations of one year bonds with bonds of up to 30 years and reporting the highest and lowest value for $R(.,.)$ in the case where $\pi^g_{HH} = \pi^g_{LL} = 0.95$ but government beliefs differ. The table suggests that our choice of one and thirty year bonds in Figure 4a was flattering to the complete market approach. Other maturities frequently lead to worse outcomes than the balance budget case when the persistence of expenditure shocks is underestimated and the losses are even greater in this direction than overestimating the persistence. Although issuing thirty year bonds is rarely the way to maximise $R(.,.)$ in the case of misperceptions it does seem that issuing such long bonds is a more robust way of minimising the losses from underestimating the persistence of shocks. It doesn't however help against the costs of overestimating persistence.

INSERT TABLE 4

To better understand the robustness of the complete market approach to model misspecification we perform the following exercise. We consider the optimal portfolios of one and thirty period debt recommended by the complete market approach when the government thinks that the persistence parameter is either 0.65, 0.75, 0.85 or 0.95 i.e four different portfolios. In the case where the government believes the persistence is 0.95 the optimal portfolio is to issue 30 period debt worth 701% of output and go short by 689% in one year bonds. The absolute size of the positions is declining in the perceived persistence such that when the government thinks the persistence parameter is 0.65 the positions are 112% and -110% respectively. We then calculate welfare for all four portfolios but where the true persistence in the economy takes values between 0.1 and 0.9. The results are shown in Figure 5. The results suggest that the balanced budget outcome is always worse than implementing complete markets under the mistaken belief that $\pi^g_{HH} = \pi^g_{LL} = 0.65$. By contrast when beliefs are that $\pi^g_{HH} = \pi^g_{LL} = 0.95$ then a balanced budget dominates nearly everywhere. The implication is that it is the size of the positions that leads to welfare losses from misspecification. Robustness consideratons would suggest reducing the magnitude of positions advocated by complete markets.

INSERT FIGURE 5 HERE

5.1.2 Misperceiving States of the World

The previous subsection focused on a minor deviation from complete markets. The government still issued enough securities (2 - the number of states of the world) to achieve the complete market outcome but because of misperceptions failed to do so. In this subsection we consider a more serious failure - the government continues to issue 2 securities but there exist three states of the world. As well as government expenditure taking on a high and a low value it can also with small probability take on a very large value, g^W(as would be the case with a war). Specifically we assume that the economy is characterised by a transition matrix $\begin{pmatrix} \pi & 1-\pi & 0 \\ 1-\pi & \pi-\pi^W & \pi^W \\ 0.05 & 0.9 & 0.05 \end{pmatrix}$ but the government perceives only a transition matrix between two states $\begin{pmatrix} \pi_{HH} & \pi_{HL} \\ \pi_{LH} & \pi_{LL} \end{pmatrix}$. Figure 6 shows the value of $R(.,.)$ as π^W varies from 0 to 0.05. For even small values of π^W there is a sharp fall in welfare from the complete market outcome due to the fact that the government ignores the possibility of a third state such that it is often optmal to follow a balanced budget rather than the complete market outcome.

INSERT FIGURE 6 HERE

5.1.3 Misperceiving the Discount Rate

The size of positions recommended by the complete market approach will depend significantly on the perceived discount rate. In this section we show how errors here again lead to it being better to use a balanced budget rather than issue debt. Consider the case where the agents discount rate is $\beta = 0.98$ but the government has beliefs in the range $(0.93, 0.98)$. Figure 7 shows the welfare performance across the various combinations. For the case of issuing a 1 and 30 period bonds any incorrect beliefs over the discount factor lead to a worse outcome than a balanced budget. This example also illustrates another non-robustness problem. We documented earlier that when governments made mistakes about the persistence of shocks there was some evidence that issuing long bonds was the most robust policy. However in the case of errors in the discount rate issuing long bonds is usually worse than the balanced budget.

INSERT FIGURE 7 HERE

5.2 Transaction Costs

In this subsection we return once more to the model of earlier in which the government has perfect knowledge but consider another problem with the size of the positions - transaction costs. In Faraglia, Marcet and Scott (2009) we show how to solve for optimal debt management under incomplete markets in the presence of transaction costs (and other market imperfections). Here however we offer some simple calculations to show the size of the problem this represents for the complete market approach. Assume the government pursues the complete market approach to debt management even when markets are incomplete. We then calculate the level of transaction costs that would make the government indifferent between pursuing this approach or a balanced budget. Specifically we solve the following problem:

$$c_t + g_t + TC = 1 - x_t$$
$$g_t + TC + (1 - p_{1,t})\, b_1 + (p_{N-1,t} - p_{N,t})\, b_N = (1 - \frac{v_{x,t}}{u_{c,t}})(1 - x_t)$$
$$p_{j,t} = \frac{\beta^j E_t\,(u_{c,t+j})}{u_{c,t}}$$
$$E_0 \sum_{t=0}^{\infty} \beta^t U_t = W_{BB}$$

In the case of government expenditure shocks only then even miniscule level of transaction costs equal to 0.016% of steady state government expenditure (or a level of transaction costs equal to 0.003% of the absolute value of debt issued) are sufficient for the government to prefer to run a balanced budget than the complete market inspired optimal debt management. In the case of both productivity and expenditure shocks then the level of transaction costs required to be indifferent with a balanced budget is 0.02% of steady state expenditure and 0.002% of the absolute value of debt issued.

We have shown in this section through a series of examples how introducing market incompleteness explicitly often produces outcomes where governments would rather avoid the insights from the complete market approach. Indeed so profound is the problem of sensitivity and volatility that governments would prefer to forego the advantages of tax smoothing completely and run a balanced budget. These results have a similar flavour to those of Siu (2004) who considers a Ramsey planner choosing optimal taxation in an economy characterised by non-state contingent debt and sticky prices. In this case the government can use fluctuations in unanticipated inflation to achieve the complete market outcome given by (7) but has to trade this off against the welfare costs of incorrect relative prices induced by sticky prices (see also Lustig, Sleet and Yeltekin (2009) for an analysis of optimal portfolio structure in the case of nominal debt and sticky prices). Siu (2004) finds that for a government expenditure process similar to the post-war US experience the required inflation volatility is so great and the associated welfare costs so large governments find it optimal not to complete the market. Only if government expenditure experiences war time spikes, essentially the fat tail model of Section 5.1.2, is it desirable to use inflation volatility to help achieve fluctuations in the market value of debt. Therefore completing the market requires extreme and volatile behaviour in a variable (in his case inflation, in ours debt positions) and because of market imperfections (for Siu sticky prices, in our case misperceptions or transaction costs) this volatility is sub-optimal.There are of course differences in our approaches. Siu focuses on an imperfection in the goods market (sticky prices) whilst our focus is on imperfections in the bond market. Further Siu is not about debt management as the government only ever issues one bond. Finally his results show that in the presence of fat tails it is optimal *ex post* to alter inflation to minimise deadweight loss whereas we find in the presence of fat tails it is not optimal *ex ante* to issue large debt positions.

6 Conclusion

Macroeconomists have become increasingly interested in trying to embed policy recommendations for debt management into theories of optimal fiscal policy This literature has produced an appealing theory which we call the complete market approach to debt management. By exploiting variations in the yield curve the government can structure its debt so as to minimize the distortionary costs to taxation. Bond price movements help maintain the government's intertemporal budget constraint that requires equating the market value of government debt to the net present value of future primary fiscal surpluses. Successful debt management enables this to happen whilst minimizing changes to taxes. The great strength of this insight is that it can be applied even in the case when bond markets are incomplete in the sense that the government cannot issue state-contingent debt. Further a number of authors have argued that this complete market approach offers a robust qualitative recommendation to debt managers - governments should issue long term debt and invest in short term bonds.

In this paper we have extensively reviewed the insights and implications of this complete market approach to debt management and identify a number of areas where this methodology is problematic:

i) As in Buera and Nicolini (2004) we find that the magnitude of the debt positions the government is required to hold are implausibly large multiples of GDP. We extend Buera and Nicolini's results by calibrating the model to US data and considering a range of extensions including capital accumulation and habits. Although the magnitude of the positions does change substantially across these model specifications they remain throughout extremely large compared with observed practice.

ii) We identify an additional problem when we extend the model to allow for capital accumulation and habits. The required positions also show extremely large volatility. In particular increasing the volatility of interest rates only partly alleviates the size of positions but introduces a problem of extreme volatility. In some cases this volatility is so large that optimal positions for long term debt fluctuate between large negative and positive positions from one period to the next.

iii) It could be argued that these defects are a result of using inevitably stylized models, that the quantitative implications of the theory should not be taken too seriously, but that the qualitative features are robust. However we find that this complete market approach is also extremely sensitive to relatively small variations in parameters. Both the size and sign of positions can change dramatically with small changes in relative persistence of shocks or slight changes in the maturity of bonds that governments can issue. There may be good reasons why governments in the real world should issue long term debt, but the complete market methodology is not what produces this recommendation.

iv) We show that by introducing varying degrees of market incompleteness these large volatile and unstable debt positions lead to sub-optimal outcomes. In particular, allowing for possible model misspecification or transaction costs we find frequently that the government would prefer to follow a balanced budget rather than implement the optimal portfolio structure recommended by the complete market approach.

The fundamental problem of the complete market approach is that the limited volatility of the yield curve makes maturities a poor substitute for state contingent debt. Therefore in order to exploit the maturity structure of debt the complete market approach requires large positions. If governments were to try and implement these policy recommendations they would have to buy and sell enormous amounts of bonds each period. This would entail all kinds of transaction costs, refinancing risks, and it would force some private agents in the economy to hold the opposite of the huge positions the government decided to take, possibly facing credit constraints. The government would have to hold very large amounts of private debt which could be defaulted upon. By explicitly ignoring these features of market incompleteness we believe the complete market approach is potentially misleading. The great strength of the complete market approach is it recognizes the importance of debt management in providing insurance against fiscal shocks. However the weakness with the complete market approach is it only focuses on fiscal insurance and abstracts from fundamental features of market incompleteness. A successful theory of debt management will need to balance the insights of fiscal insurance with the constraints that incomplete markets provide. Whilst the complete market approach offers many insights we do not think it can be used to justify debt management policies or recommendations. We remain in search of a plausible theory of debt management.

APPENDIX A - Solution Details

Here we present the equations determining the equilibrium and we describe in detail the numerical computations for each model analyzed. In all cases it was assumed that initial government debt was zero, so we describe the solution procedure for this case, guaranteeing that there is no difference between the policy function of period zero and all other periods.

Numerical solution of the endowment economy (Section 2)

When $b_{-1} = 0$ the Lagrangian of the Ramsey problem of the endowment economy is:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \{U(c_t) + V(x_t) + \lambda [U_{c,t} c_t - V_{x,t}(1 - x_t)] \\ &\quad + \nu_t [\theta_t (1 - x_t) - c_t - g_t]\} \end{aligned} \tag{22}$$

The first order conditions of the problem for all t are

$$\begin{aligned} U_{c,t} + \lambda (U_{cc,t} c_t + U_{c,t}) - \nu_t &= 0 \\ V_{x,t} - \lambda (V_{xx,t}(1 - x_t) - V_{x,t}) + \nu_t &= 0 \\ \theta_t (1 - x_t) - c_t - g_t &= 0 \end{aligned} \tag{23}$$

plus the implementability constraint (5).

Moreover assume that the shocks follow a Markov process of N^2 states, $\{\overline{g}_i, \overline{\theta}_j\}$ with $i, j = 1, ..., N$.

For a given value of λ solving the model is trivial: for each period and each realization equations (23) give three equations to find the three unknowns c_t, x_t, ν_t as a (time-invariant) function of the exogenous shocks θ, g. In order to find the equilibrium λ we perform the following steps:

1. given the initial condition and the transition probabilities of the states, draw S series of T periods each of the shocks θ, g using a random number generator. Denote this realization $\left\{ \{g_t^i, \theta_t^i\}_{t=1}^T \right\}_{i=1}^S$. The number of series S should be large enough for a certain expectation that we specify below to be computed accurately. T should be large enough for a certain discounted sum that we describe below to be computed accurately.

2. guess a value for λ. Solve system (23) for every state$\{\overline{g}_i, \overline{\theta}_j\}$ to get N^2 values of c, x and the corresponding surplus;

3. given the values from 2., for a given realization $\left\{g_t^i, \theta_t^i\right\}_{t=1}^T$ we could approximate the discounted with the sum $\sum_{t=0}^{T} \beta^t \left(c_t^i \frac{U_{c,t}^i}{U_{c,0}^i} - (1 - x_t^i) \frac{V_{x,t}^i}{U_{c,0}^i}\right)$. This amounts to setting all surpluses for $t > T$ equal to zero, and we can do better than that. We can reduce the error from truncating the sum by setting the surplus for $t > T$ to the average value of the deficit for each λ. So, to the truncated sum we add $\frac{1-\beta^T}{1-\beta} \left(\overline{c} \frac{\overline{U}_c}{U_{c,0}} - (1 - \overline{x}) \frac{\overline{V}_x}{U_{c,0}}\right)$ where $\overline{U}_c, \overline{c}, \overline{x}$ and $\overline{V}_x$ are computed at the mean of the shocks.

   Finally, compute the average of discounted sums

$$\frac{1}{S} \sum_{i=1}^{S} \left[\sum_{t=0}^{T} \beta^t \left(c_t \frac{U_{c,t}}{U_{c,0}} - (1 - x_t) \frac{V_{x,t}}{U_{c,0}}\right)\right] + \frac{1 - \beta^{T+1}}{1 - \beta} \left(c_T \frac{\overline{U}_{c,T}}{U_{c,0}} - (1 - x_T) \frac{\overline{V}_{x,T}}{U_{c,0}}\right) \qquad (24)$$

   which should be a good approximation to

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(c_t \frac{U_{c,t}}{U_{c,0}} - (1 - x_t) \frac{V_{x,t}}{U_{c,0}}\right)$$

   for S, T sufficiently large. Notice that (24) is a function of λ.

4. Iterate on λ until (24) is close to zero. The result is the equilibrium λ

Given this equilibrium λ and values of c, x and of the surpluses, we compute the prices in all the states of the bonds with different maturities, computing the expectation on marginal utilities as a simple sum over all possible future states.

We also can compute the z's by a regression of the realized discounted sum on the exogenous variables.

For every maturity we can calculate the value of the matrix of returns and compute:

$$b = Pz$$

where $\underset{(N^2 \times 1)}{b}$ is the vector of bonds, $\underset{(N^2 \times N^2)}{P}$ is the matrix of the returns and $\underset{(N^2 \times 1)}{z}$ is the vector of conditional expected discounted surpluses.

Numerical solution of the economy with capital (Section 3)

Even with zero initial debt the model with capital has a different policy function in period zero from the following periods. The reason is that the return to capital is in the right side of the implementability constraint.

Assuming $\tau_0^k = 0$, the Lagrangian of the Ramsey problem is:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \{U(c_t) + V(x_t) + \lambda [U_{c,t} c_t - V_{x,t}(1 - x_t)] \\ &\quad + \nu_t [F(k_{t-1}, 1 - x_t, \theta_t) + (1 - \delta) k_{t-1} - c_t - g_t - k_t] \\ &\quad - \lambda [b_{-1} + (F_{k,0} + 1 - \delta) k_{-1}] U_{c,0}\} \end{aligned}$$

and the first order conditions are:

for $t > 0$:

$$\begin{aligned} U_{c,t} + \lambda (U_{cc,t} c_t + U_{c,t}) - \nu_t &= 0 \\ V_{x,t} - \lambda (V_{xx,t}(1 - x_t) - V_{x,t}) + \nu_t F_{x,t} &= 0 \\ \nu_t - \beta E_t [\nu_{t+1}(F_{k,t+1} + 1 - \delta)] &= 0 \\ F(k_{t-1}, 1 - x_t, \theta_t) + (1 - \delta) k_{t-1} - c_t - g_t - k_t &= 0 \end{aligned} \tag{25}$$

for $t = 0$:

$$\begin{aligned} U_{c,0} + \lambda (U_{cc,0} c_0 + U_{c,0}) - \nu_0 - \lambda [b_{-1} + (F_{k,0} + 1 - \delta) k_{-1}] U_{cc,0} &= 0 \\ V_{x,0} - \lambda (V_{xx,0}(1 - x_0) - V_{x,0}) + \nu_0 F_{x,0} - \lambda F_{kx,0} k_{-1} &= 0 \\ \nu_0 - \beta E_0 [\nu_1 (F_{k,1} + 1 - \delta)] &= 0 \\ U_{c,0} - \beta E_0 \left[U_{c,1} \left(\tau_1^k F_{k,1} + 1 - \delta\right)\right] &= 0 \\ F(k_{-1}, 1 - x_0, \theta_0) + (1 - \delta) k_{-1} - c_0 - g_0 - k_0 &= 0 \end{aligned} \tag{26}$$

We assume log utility and $b_{-1} = 0$.

The numerical procedure that we follow has step 1) as above. The following steps are now a bit more involved:

1. guess a value for λ. Given results in Chari, Christiano and Kehoe (1994) the solution after period 1 is given by a time-invariant function of the state variables k_{t-1}, g_t, θ_t, so we parameterize the function

$$E_t [U_{c,t+1}(F_{k,t+1} + 1 - \delta)] = \Phi(\beta; k_{t-1}, g_t, \theta_t), \text{ for } t \geq 1$$

   where Φ is a polynomial with parameters β.

2. Given the assumption of log utility the first equation in (25) gives $U_{c,t} = \nu_t$ and the third equation in (25) gives

$$U_{c,t} = \Phi\left(\beta; k_{t-1}, g_t, \theta_t\right)$$

Given Φ and a conjecture for β we draw a long realization (10000 periods) of the shocks and we use system (25) to generate long run simulations for all variables and iterate on β with PEA (den Haan and Marcet (1990)) to find the fixed point β_f. In this way we find an approximation to the policy function for $t > 0$ consistent with λ.

3. period 0 is different from the other periods. Now $U_{c,0} \neq \nu_0$. To find the optimal choice for period 0, guess a value for k_0. For every value of g_1, θ_1 solve period 1 variables using system (25) replacing E_1 by the approximate function found in the previous step. Averaging over all states for g_1, θ_1 compute $E_0\left[\nu_1\left(F_{k,1} + 1 - \delta\right)\right]$, $E_0\left(U_{c,1}F_{k,1}\right)$, and $E_0\left(U_{c,1}\right)$ consistent with each k_0. Finally, solve the system (26) for the period $t = 0$ variables, setting $\tau_1^k = \left(1 - \frac{U_{c,0}}{\beta E_0\left(U_{c,1}F_{k,1}\right)} + (1-\delta)\frac{E_0\left(U_{c,1}\right)}{E_0\left(U_{c,1}F_{k,1}\right)}\right)$, the level of capital tax that satisfies the first order conditions of the consumer.

4. perform a long simulation (100000 periods) of the model given k_0 found in step 3. and using $\Phi\left(\beta_f; k_{t-1}, g_t, \theta_t\right)$ for the remaining periods, given the realization for (c_t, x_t, k_t) from point 4), we approximate the infinite sum of the surpluses as a function of the states:

$$E_t \sum_{j=t+1}^{\infty} \beta^{j-t}\left\{U_{c,j}c_j + V_{x,j}\left(1 - x_j\right)\right\} = \Omega\left(\widetilde{\beta}_f; k_{t-1}, g_t, \theta_t\right);$$

by constructing the infinite sums in the expectation and running one regression of that infinite sum on $\Omega\left(\widetilde{\beta}_f; k_{t-1}, g_t, \theta_t\right)$. This is used to reduce the error in truncating the infinite sum as in step 3 of the previous model.

5. short simulation: we draw 10000 realizations of the shocks for the first 50 periods. We solve (25) given k_0 and we compute the infinite sum of the expected surplus in period 0 as an average of the infinite sums using the short simulations for the first 50 periods and $\Omega\left(\widetilde{\beta}_f; k_{t-1}, g_t, \theta_t\right)$ for $t = 51$; to approximate the left side of the budget constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t\left[U_{c,t}c_t - V_{x,t}\left(1 - x_t\right)\right] = \left[b_{-1} + \left(F_{k,0} + 1 - \delta\right)k_{-1}\right]U_{c,0}$$

and we iterate on λ in a similar way as before, until this constraint is approximately satisfied.

Given the equilibrium λ, Φ, β_f and the realizations of (c_t, x_t, k_t) of the long simulation of point 4), in order to get the bond prices at different maturities (from 1 to 30 years), approximate the expectations of future marginal utilities as a function of the current states of the economy.

Select 10000 consecutive periods of the long simulation. The vector of D's in (18) is given by $D(k_{t-1}(h^{t-1}), \overline{g}_i, \overline{\theta}_j) = \Omega\left(\widetilde{\beta}_f; k_{t-1}(h^{t-1}), \overline{g}_i, \overline{\theta}_j\right) + \left[U_{c,t}c_t + V_{x,t}\left(1 - x_t\right)\right]\left(h^{t-1}, \overline{g}_i, \overline{\theta}_j\right)$, the prices Π are computed in a similar way, and (18) gives the equilibrium maturities for the one period bond and different maturities of the longer bonds.

Numerical solution of the economy with consumption habits (Section 4)

In the case of habits the Lagrangian of the Ramsey problem with zero initial debt is exactly as in (22) replacing $U(c_t)$ by $U(c_t, c_{t-1})$ and with $U_{c,t}$ given by (20). This means that in the discounted sum of the Lagrangian future consumptions appear in the term dated t, so that in order to formulate the model recursively we have to rearrange the Lagrangian

For this purpose, use the notation

$$
\begin{aligned}
U_{1,t} &= \frac{\partial\, U(c_t, c_{t-1})}{\partial c_t} \qquad U_{2,t} = \frac{\partial\, U(c_t, c_{t-1})}{\partial c_{t-1}} \\
U_{11,t} &= \frac{\partial^2\, U(c_t, c_{t-1})}{\partial (c_t)^2} \qquad U_{22,t} = \frac{\partial^2\, U(c_t, c_{t-1})}{\partial (c_{t-1})^2} \qquad U_{12,t} = \frac{\partial^2\, U(c_t, c_{t-1})}{\partial c_{t-1} \partial c_t}
\end{aligned}
\tag{27}
$$

With this notation $U_{c,t} = U_{1,t} + \beta\ E_t U_{2,t+1}$ and the Lagrangian can be written as

$$
\begin{aligned}
L &= E_0 \sum_{t=0}^{\infty} \beta^t\ \{U(c_t, c_{t-1}) + V(x_t) + \lambda\ \left[(U_{1,t} + \beta\ U_{2,t+1})\ c_t - V_{x,t}(1 - x_t)\right] \\
&\quad + \nu_t \left[\theta_t (1 - x_t) - c_t - g_t\right]\} \\
&= U(c_0, c_{-1}) + V(x_0) + \lambda\ \left[U_{1,0}\ c_0 - V_{x,0}(1 - x_0)\right]\ + \nu_0 \left[\theta_0 (1 - x_0) - c_0 - g_0\right] \\
&\quad + E_0 \sum_{t=1}^{\infty} \beta^t\ \{U(c_t, c_{t-1}) + V(x_t) + \lambda\ \left[U_{1,t}\ c_t + U_{2,t}\ c_{t-1} - V_{x,t}(1 - x_t)\right] \\
&\quad + \nu_t \left[\theta_t (1 - x_t) - c_t - g_t\right]\}
\end{aligned}
$$

where we used the notation in (27) and the the law of iterated expectations in the first equality, and for the last equality we set aside the $t = 0$ term and reorder.

Notice that the term multiplying λ is different in period $t = 0$ as in future periods so that the solution is only recursive after $t > 0$. This means that even with zero initial debt the policy function is different in the first period, unlike the endowment model of section 2, but similar to the case with capital. Notice also that c_{t-1} is the only variable from the past that appears in the terms dated $t \geq 1$ so that this is a sufficient state variable. This implies that the optimal solution can be written recursively as $c_t = G(h_t, c_{t-1})$ for $t > 0$ but a different decision function applies at time zero.

The FOC for $t > 0$ are given by the following expression

$$(1+\lambda)\left(U_{1,t} + \beta E_t U_{2,t+1}\right) + \lambda\left[U_{11,t}c_t + U_{12,t}\ c_{t-1} + \beta E_t\left(U_{12,t+1}c_{t+1}\right) + \beta\ c_t\ E_t U_{21,t+1}\right] = \nu_t$$

but the term $U_{12,t}\ c_{t-1}$ is absent for the FOC at t=0.

To take care of the different first order condition for period zero we compute c_0 from an analog to step 3 in the algorithm described for the model of section 3.

Notice that the above FOC imply that the expectations $E_t U_{2,t+1}$, $E_t\left(U_{12,t+1}c_{t+1}\right)$ and $E_t U_{21,t+1}$ need to be approximated in order to use this FOC to solve for c_t. Given the functional form for U used in the simulations we have

$$\begin{aligned} E_t U_{2,t+1} &= -\chi E_t\left[(c_{t+1} - \chi c_t)^{-\gamma_1}\right] \\ E_t U_{12,t+1} &= \chi\gamma_1\ E_t\left[(c_{t+1} - \chi c_t)^{-\gamma_1 - 1}\right] \\ E_t\left(U_{12,t+1}c_{t+1}\right) &= \chi\gamma_1\ E_t\left[c_{t+1}(c_{t+1} - \gamma c_t)^{-\gamma_1 - 1}\right] \end{aligned}$$

We proceed by parameterizing three expectations

$$\begin{aligned} E_t\left[(c_{t+1} - \gamma c_t)^{-\gamma_1}\right] &= \Phi_1\left(\beta^1; c_{t-1}, g_t, \theta_t\right) \\ E_t\left[(c_{t+1} - \gamma c_t)^{-\gamma_1 - 1}\right] &= \Phi_2\left(\beta^2; c_{t-1}, g_t, \theta_t\right) \\ E_t\left[c_{t+1}(c_{t+1} - \gamma c_t)^{-\gamma_1 - 1}\right] &= \Phi_3\left(\beta^3; c_{t-1}, g_t, \theta_t\right) \end{aligned}$$

Then we solve for a rational expectations equilibrium given λ, we compute the initial consumption separately, check the value of the implementability constraint for each λ, and iterate on λ until the implementability constraint is satisfied.

We build the elements of the system of equations that give the bonds at each maturity by approximating the corresponding functions of future discounted deficits and prices, for each value of the state variable c_t and for each possible future realization, now these functions have to depend on past consumption.

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Table 1 : Simulation Results - Endowment economy

Shocks							Interest rates			
g							**H**	**L**		
		B_1	B_{30}			R_1	2.23	1.85		
	$\mu = 1$	-7.04	7.16			R_{30}	2.10	1.98		
		B_1	B_{30}			R_1	3.95	0.13		
	$\mu = 0$	-0.79	0.81			R_{30}	2.28	1.80		
$\boldsymbol{\theta}$							**H**	**L**		
		B_1	B_{30}			R_1	1.07	2.93		
	$\mu = 1$	-0.85	0.90			R_{30}	1.85	2.21		
		B_1	B_{30}			R_1	-3.13	7.21		
	$\mu = 0$	-0.17	0.18			R_{30}	1.86	2.21		
$\mathbf{g}, \boldsymbol{\theta}$							**HH**	**HL**	**LH**	**LL**
		B_1	B_4	B_{13}	B_{30}	R_1	1.23	3.25	0.90	2.71
	$\mu = 1$	-16.15	41.32	-86.71	57.66	R_{30}	1.92	2.28	1.79	2.15
$\pi^g_{HH} = 0.95$		B_1	B_2	B_3	B_{29}	R_1	-5.75	7.21	-2.98	4.16
$\pi^\theta_{HH} = 0.91$	$\mu = 1/3$	-4.22	58.48	-161.22	106.37	R_{29}	1.92	2.28	1.79	2.14
		B_1	B_5	B_{18}	B_{30}	R_1	2.00	2.45	1.64	2.71
	$\mu = 1$	63.82	-140.94	163.15	-75.64	R_{30}	1.97	2.22	1.85	2.09
$\pi^g_{HH} = 0.95$		B_1	B_2	B_3	B_{29}	R_1	-3.34	6.96	-2.74	4.02
$\pi^\theta_{HH} = 0.98$	$\mu = 1/3$	5.77	-85.8	210.19	-129.51	R_{29}	1.91	2.28	1.79	2.14

Table 2 : Simulation Results - Capital Accumulation

Shocks							**Interest rates**			
							H	**L**		
g		B_1	B_{30}							
	$\mu = 1$	-14.49	12.36			R_1	2.08	1.98		
	$E_{+5\%}$	-18.29	9.41			R_{30}	2.07	2.00		
	$E_{-5\%}$	-11.65	16.3							
		B_1	B_{30}							
	$\mu = 0$	-9.23	7.19			R_1	2.06	1.99		
	$E_{+5\%}$	-9.50	6.90			R_{30}	2.04	2.03		
	$E_{-5\%}$	-8.94	7.46							
$\boldsymbol{\theta}$							**H**	**L**		
		B_1	B_{30}							
	$\mu = 1$	-8.49	6.26			R_1	2.26	1.85		
	$E_{+5\%}$	-12.5	3.56			R_{30}	2.01	2.07		
	$E_{-5\%}$	-5.62	10.10							
		B_1	B_{30}							
	$\mu = 0$	-3.49	1.47			R_1	2.01	2.07		
	$E_{+5\%}$	-3.93	1.19			R_{30}	2.02	2.06		
	$E_{-5\%}$	-3.12	1.82							
$\mathbf{g}, \boldsymbol{\theta}$							**HH**	**HL**	**LH**	**LL**
		B_1	B_4	B_{16}	B_{30}					
	$\mu = 1$	-30.10	42.54	-48.18	33.44	R_1	2.46	1.67	2.26	1.48
	$E_{+5\%}$	-34.33	26.14	-97.58	15.94	R_{30}	2.03	2.07	1.94	1.98
	$E_{-5\%}$	-26.30	63.28	-16.46	66.29					
$\pi_H^g = 0.95$		B_1	B_9	B_{13}	B_{29}					
$\pi_H^\theta = 0.91$	$\mu = 1/3$	-14.38	32.62	-30.74	10.42	R_1	2.04	1.97	2.00	1.92
	$E_{+5\%}$	-18.80	26.24	-36.75	8.16	R_{29}	2.01	2.05	2.00	2.03
	$E_{-5\%}$	-11.00	41.44	-25.37	11.91					
		B_1	B_5	B_{18}	B_{30}					
	$\mu = 1$	-77.85	153.10	-207.77	130.19	R_1	2.55	1.63	2.42	1.50
	$E_{+5\%}$	-109.15	138.74	-226.37	106.12	R_{30}	2.09	2.05	2.02	1.99
	$E_{-5\%}$	-55.63	167.34	-189.63	161.17					
$\pi_H^g = 0.95$		B_1	B_9	B_{14}	B_{29}					
$\pi_H^\theta = 0.98$	$\mu = 1/3$	-12.58	21.44	-23.13	12.20	R_1	2.07	1.94	2.03	1.90
	$E_{+5\%}$	-34.93	13.46	-54.90	8.63	R_{29}	2.03	2.00	2.05	2.01
	$E_{-5\%}$	-5.48	70.24	-18.56	17.44					

Note: The positions and the interest rates are obtained as average of 10000 period simulation. $E_{\pm 5\%}$ denote the average conditional on the realization being among the highest or lowest 5% values of the bonds.

Table 3: Simulation Results - with consumption habits

Shocks	Habits							Interest rates			
$\boldsymbol{\theta}$								**H**	**L**		
	$\chi = 0$		B_1	B_{10}			R_1	1.07	2.93		
		$\mu = 1$	-1.03	1.07			R_{10}	1.58	2.47		
	$\chi = 0.273$		B_1	B_{10}							
		$\mu = 1$	-0.63	0.62			R_1	-0.58	5.10		
		$E_{+5\%}$	-0.68	0.59			R_{10}	1.37	2.73		
		$E_{-5\%}$	-0.58	0.66							
$\mathbf{g}, \boldsymbol{\theta}$								**HH**	**HL**	**LH**	**LL**
	$\chi = 0$		B_1	B_{10}	B_{16}	B_{30}	R_1	1.23	3.15	0.90	2.71
		$\mu = 1$	-4.60	71.74	-159.02	101.39	R_{30}	1.92	2.28	1.79	2.15
$\pi^g_{HH} = 0.95$											
$\pi^\theta_{HH} = 0.91$	$\chi = 0.25$		B_1	B_{10}	B_{15}	B_{22}					
		$\mu = 1$	-0.48	-18.23	7.01	11.48	R_1	0.09	5.39	-0.77	3.50
		$E_{+5\%}$	-0.50	-27.45	-91.14	-62.69	R_{22}	1.82	2.44	1.62	2.21
		$E_{-5\%}$	-0.45	-7.24	90.36	99.10					

Note: idem as previous table. In the model with consumption habits and technology shocks χ is chosen to match the one step ahead forecast error of the variable $spread_t + 9rr_t^{10}$ where $spread_t = rr_t^{10} - rr_t^1$. In the model with government spending and techology shocks χ is chosen to match the one spep ahead forecast error of $spread_t$.

Table 4:

$\widetilde{\pi}$	max R	R	min R
0.99	-1.384	-3.319	-8.290
0.98	0.377	-0.067	-2.740
0.97	0.841	0.750	-0.026
0.96	0.974	0.962	0.818
0.95	1.000	1.000	1.000
0.94	0.987	0.982	0.727
0.93	0.959	0.944	-0.434
0.92	0.927	0.901	-3.269
0.91	0.894	0.857	-8.840
0.90	0.863	0.815	-17.998
0.89	0.834	0.776	-30.565
0.88	0.807	0.739	-44.899
0.87	0.782	0.705	-58.576
0.86	0.760	0.674	-69.745
0.85	0.739	0.645	-77.849
0.84	0.720	0.618	-83.321
0.83	0.703	0.593	-86.930
0.82	0.687	0.570	-89.365
0.81	0.672	0.548	-91.104

Figure 1: Sensitivity of Portfolio Structure - Capital Accumulation and Persistent Technology Shocks

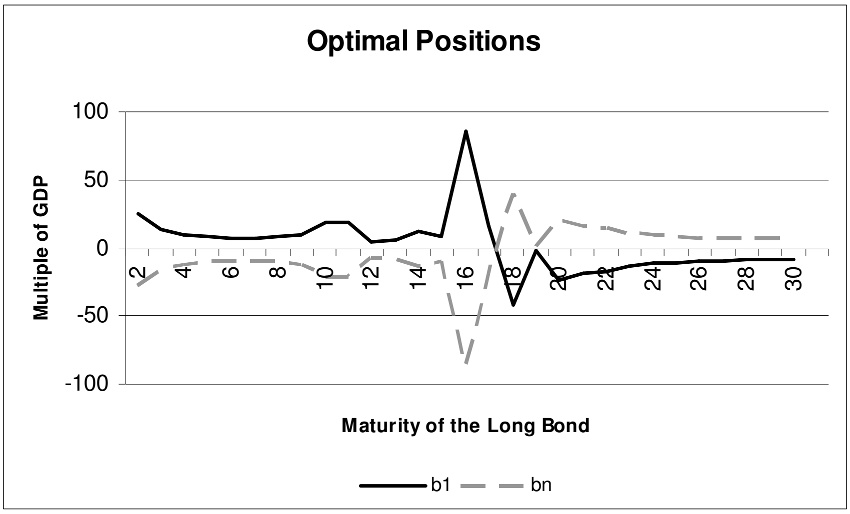


Figure 2: Policy Functions for Debt Issuance - Capital Accumulation and Persistent Technology Shocks

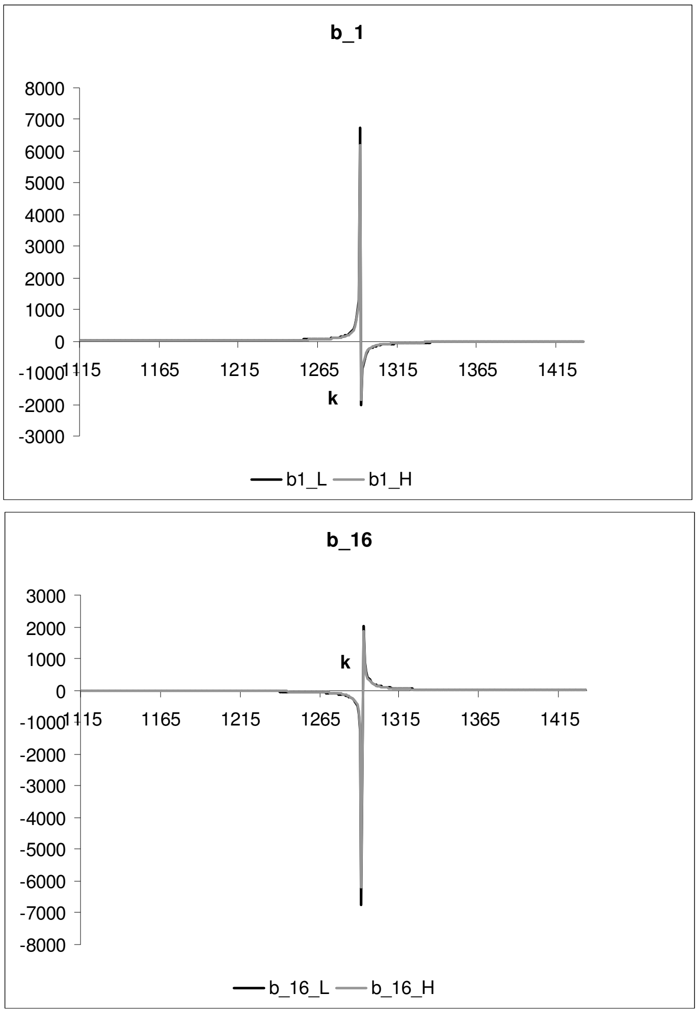


Figure 3 : Policy Functions for Debt Issuance - Habits and Both Shocks

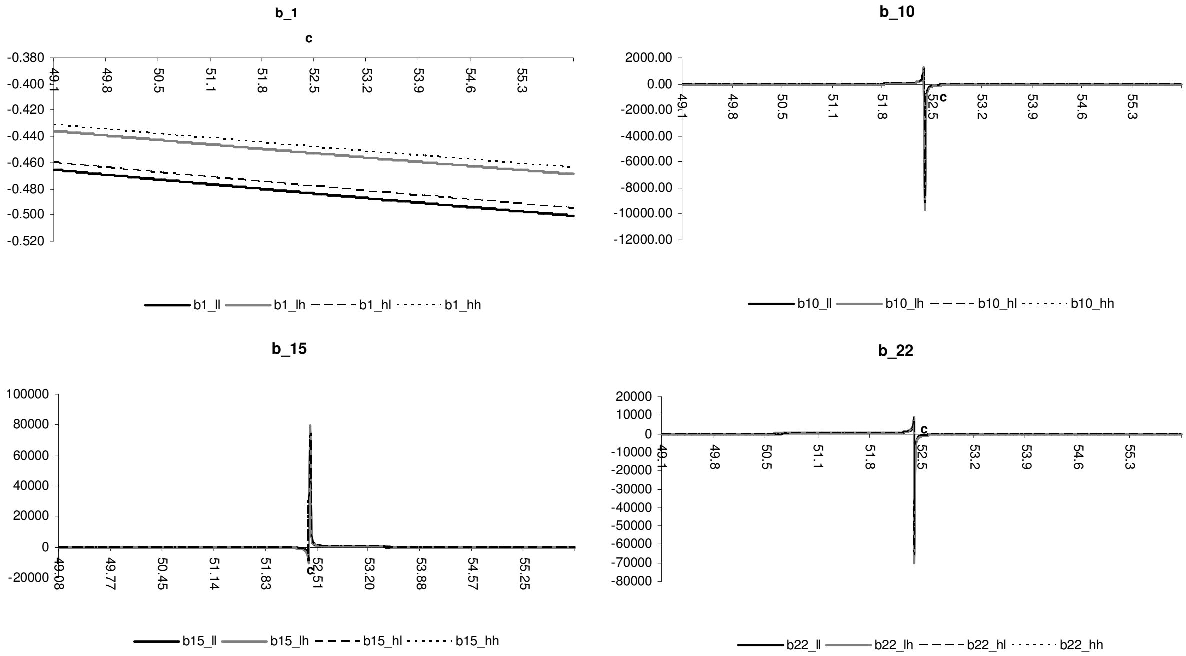


Figure 4a : Uncertainty in the transition probabilities: 2 state economy - $\pi = 0.95$

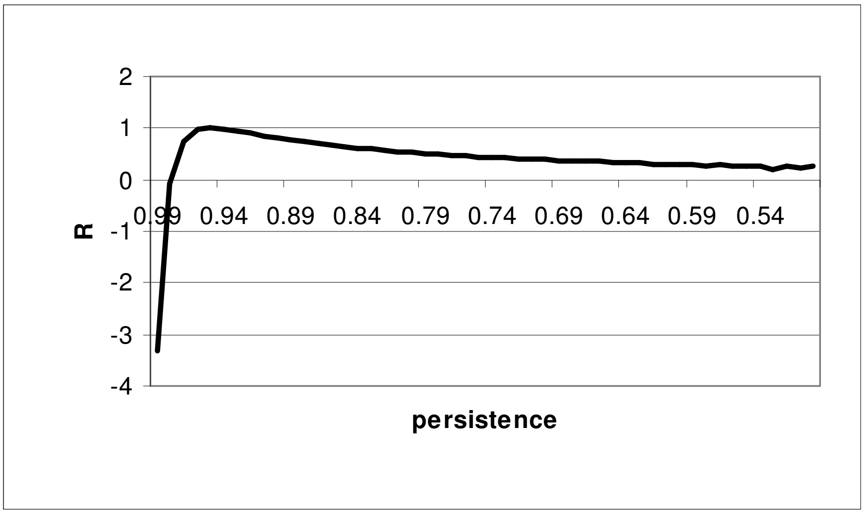


Figure 4b : Uncertainty in the transition probabilities: 2 state economy - $\pi = 0.5$

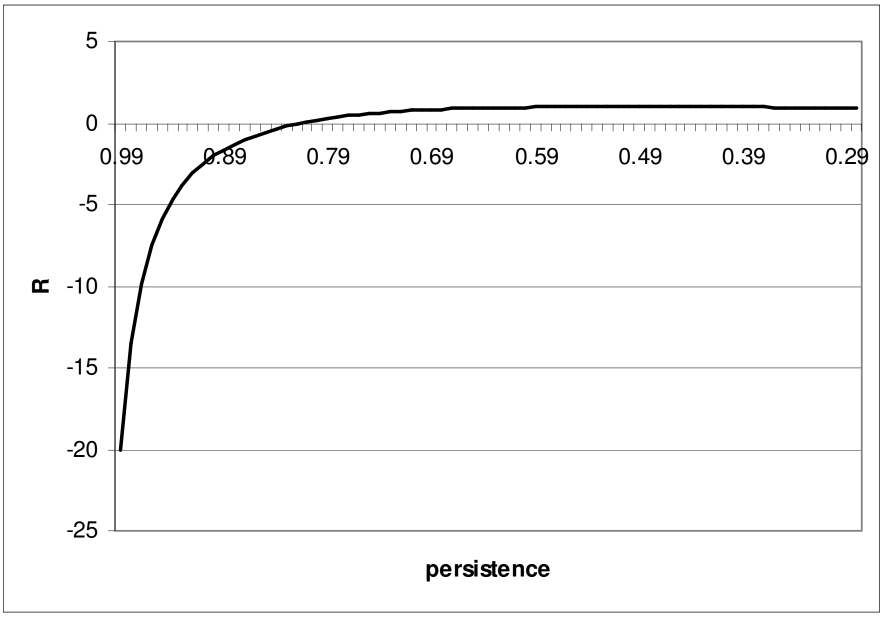


Figure 5: Horse race: Balanced Budget (BB) against ABN

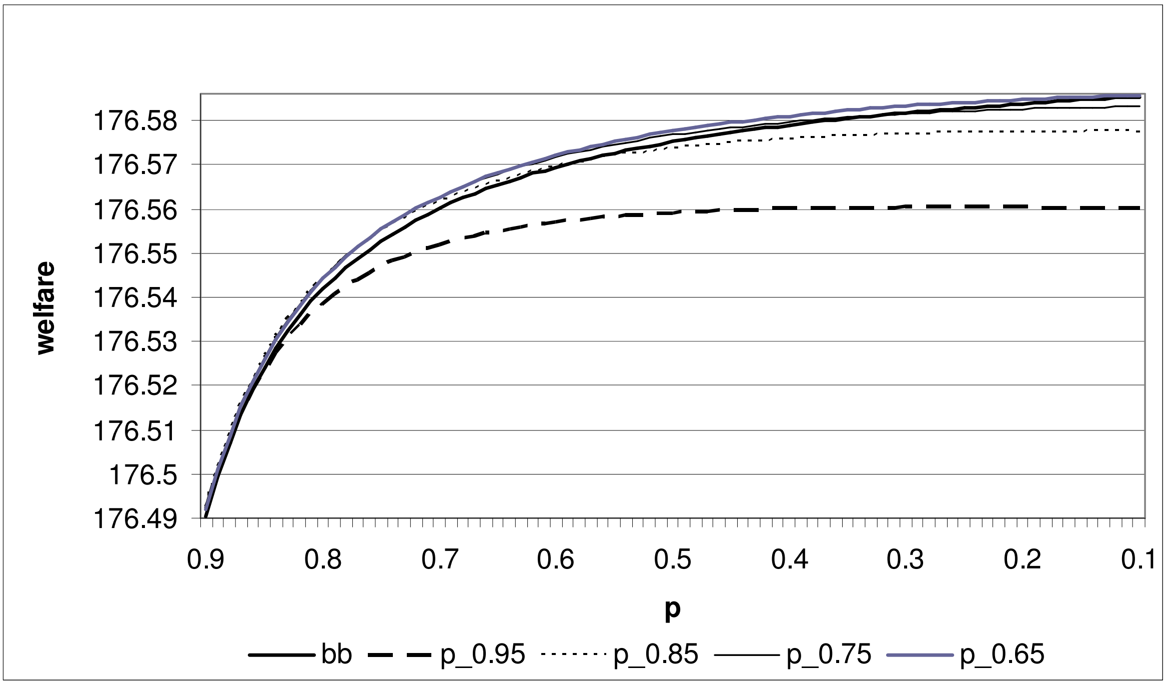


Figure 6: Hidden State - 3 state economy but 2 bonds

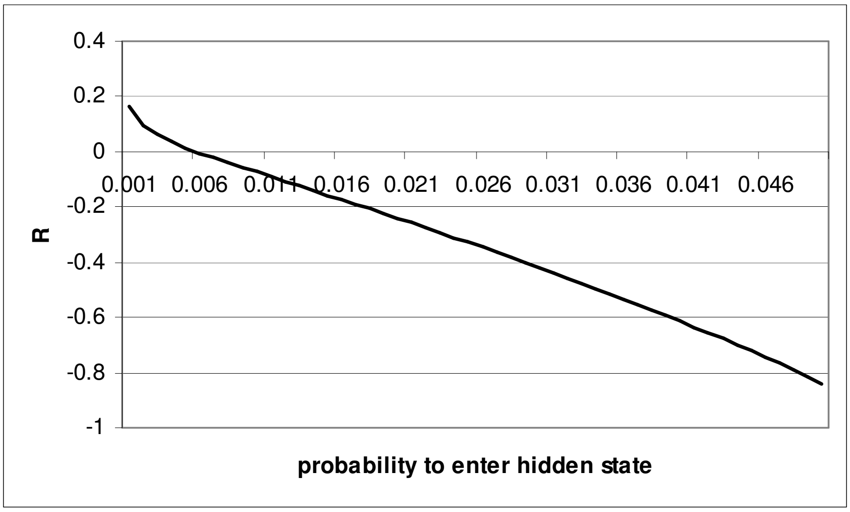
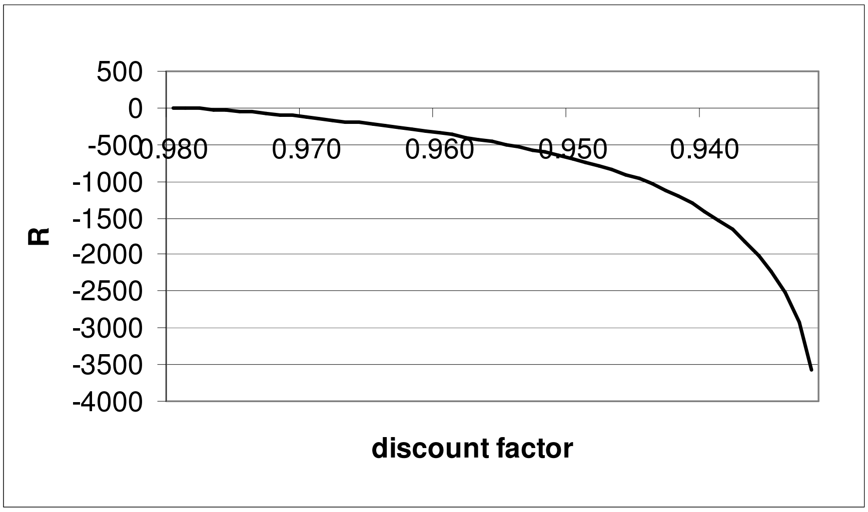


Figure 7: Uncertainty in the discount factor: 2 state economy



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